# Worksheet 9, Math 1B

## Taylor and Maclaurin Series

Monday, March 12, 2012

1. Find the Taylor series for $f(x) = x^4 - 3x^2 + 1$, centered at $a = 1$ and $a = 0$.
2. Find the Taylor series for $f(x) = e^x$, centered at $a = 3$.
3. Find the Maclaurin series for $e^x + e^{2x}$.
4. Find the Maclaurin series for $\cosh(x)$ by manipulating known series. Compare the series you find with that for $\cos(x)$.
5. How many terms of the Maclaurin series for $\sin x$ do you need to add together in order to compute $\sin 3^\circ$ correct to five decimal places?
6. Find the Maclaurin series for $\sin^{-1} x$. [Hint: Consider the Maclaurin series for $\frac{\mathrm{d}}{\mathrm{d}x} \sin^{-1} x$.]

# Worksheet 9 Solutions, Math 1B

## Taylor and Maclaurin Series

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1. Find the Taylor series for $f(x) = x^4 - 3x^2 + 1$, centered at $a = 1$ and $a = 0$.

   *Solution*

   We have derivatives:

   | Function | At $x$ | At 1 | At 0 |
   |---|---|---|---|
   | $f$ | $x^4 - 3x^2 + 1$ | -1 | 1 |
   | $f'$ | $4x^3 - 6x$ | -2 | 0 |
   | $f''$ | $12x^2 - 6$ | 6 | -6 |
   | $f^{(3)}$ | $24x$ | 24 | 0 |
   | $f^{(4)}$ | 24 | 24 | 24 |
   | $f^{(n)}$ $(n \geq 5)$ | 0 | 0 | 0 |

   Then the Taylor series centered at $a = 1$ is given by

   $$\begin{aligned} &\frac{-1}{0!} + \frac{-2(x-1)}{1!} + \frac{6(x-1)^2}{2!} + \frac{24(x-1)^3}{3!} + \frac{24(x-1)^4}{4!} \\ &= -1 - 2(x-1) + 3(x-1)^2 + 6(x-1)^3 + (x-1)^4. \end{aligned}$$

   Likewise, the Taylor series centered at $a = 0$ is given by

   $$\frac{1}{0!} + \frac{-6x^2}{2!} + \frac{24x^4}{4!} = 1 - 3x^2 + x^4.$$

   Notice that the Taylor series centered at $a = 0$ is just the original polynomial. If you multiply out the Taylor series centered at $a = 1$, it will also be equal to the original polynomial. This is because the Taylor approximation is an approximation by successively higher degree polynomials, and so using a polynomial for $f$ means that the approximations actually become exact once they reach the corresponding degree of $f$.

2. Find the Taylor series for $f(x) = e^x$, centered at $a = 3$.

   *Solution*

   All derivatives of $e^x$ are just $e^x$, so $f^{(n)}(3) = e^3$ for each $n$. This gives us a Taylor series of

   $$\sum_{n=0}^{\infty} \frac{e^3}{n!}(x-3)^n.$$

3. Find the Maclaurin series for $e^x + e^{2x}$.

   *Solution*

   Adding the corresponding series for the summands gives us

   $$\sum_{n=0}^{\infty} \frac{x^n}{n!} + \sum_{n=0}^{\infty} \frac{(2x)^n}{n!} = \sum_{n=0}^{\infty} \frac{1+2^n}{n!} x^n.$$

4. Find the Maclaurin series for $\cosh(x)$ by manipulating known series. Compare the series you find with that for $\cos(x)$.

   *Solution*

   We have that $\cosh(x) = 1/2 \cdot (e^x + e^{-x})$, so by combining the Maclaurin series for $e^x$ and $e^{-x}$, we have

$$\frac{1}{2} \cdot \left( \sum_{n=0}^{\infty} \frac{x^n}{n!} + \sum_{n=0}^{\infty} \frac{(-x)^n}{n!} \right) = \sum_{n=0}^{\infty} \frac{1 + (-1)^n}{2n!} x^n = \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!}.$$

   In particular, we notice that this is exactly the same as the series for $\cos(x)$, except that it is not an alternating series.

5. How many terms of the Maclaurin series for $\sin x$ do you need to add together in order to compute $\sin 3^\circ$ correct to five decimal places?

   *Solution Sketch*

   The Maclaurin series for $\sin x$ is an alternating series with terms decreasing to 0, and so we can apply the alternating series estimation theorem to find an error bound less than $5 \cdot 10^{-6}$, which will give us five decimal places of accuracy upon rounding. It is necessary to convert $3^\circ$ into radians before making the approximation: $3^\circ \cdot (2\pi \text{ radians})/360^\circ$.

6. Find the Maclaurin series for $\sin^{-1} x$. [Hint: Consider the Maclaurin series for $\frac{\mathrm{d}}{\mathrm{d}x} \sin^{-1} x$.]

   *Solution Sketch*

   We have that $\frac{\mathrm{d}}{\mathrm{d}x} \sin^{-1} x = 1/\sqrt{1 - x^2}$, so using a binomial series expansion, we have that this derivative is equal to

$$\sum_{n=0}^{\infty} (-1)^n \binom{-1/2}{n} x^{2n}$$

   for $|x| < 1$. Since this is the Maclaurin series for the derivative of $\sin^{-1}$, we can find the corresponding Maclaurin series for $\sin^{-1}$ by integrating, which gives us the series

$$\sum_{n=0}^{\infty} \frac{(-1)^n \binom{-1/2}{n}}{2n+1} x^{2n+1} + C.$$

   Solving for $C$ by plugging in $x = 0$ and comparing with $\sin^{-1}(0)$ gives us $C = 0$, and so we have that

$$\sin^{-1}(x) = \sum_{n=0}^{\infty} \frac{(-1)^n \binom{-1/2}{n}}{2n+1} x^{2n+1}$$

   for $|x| < 1$.