Parametric Particle Motion (BC Only)

Particle motion problems on the AP Calculus BC exam are often in the context of parametric equations or in the context of vectors.

Suppose that a particle has a position vector given by $(x(t), y(t))$ at time t.

- **Velocity:** $v(t) = (x(t), y'(t)) = \left(\frac{dx}{dt}, \frac{dy}{dt}\right)$
- **Speed (a.k.a, magnitude of velocity):** $|v(t)| = \sqrt{(x'(t))^2 + (y'(t))^2} = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}$
- **Acceleration:** $a(t) = (x''(t), y''(t)) = \left(\frac{d^2x}{dt^2}, \frac{d^2y}{dt^2}\right)$
- **Distance Traveled between** $\boldsymbol{t = a}$ **and** $\boldsymbol{t = b}$**:** $\int_a^b \sqrt{(x'(t))^2 + (y'(t))^2}\, dt = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$
- **Parametric Definition of Slope:** $\frac{dy}{dx} = \frac{y'(t)}{x'(t)} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$
- **Parametric Interpretations of Particle Motion:**
  - $\frac{dx}{dt} < 0 \iff$ The particle is moving left
  - $\frac{dx}{dt} > 0 \iff$ The particle is moving right
  - $\frac{dy}{dt} < 0 \iff$ The particle is moving down
  - $\frac{dy}{dt} > 0 \iff$ The particle is moving up
  - $\frac{dx}{dt} = 0 \iff$ The particle's position graph has a vertical tangent
  - $\frac{dy}{dt} = 0 \iff$ The particle's position graph has a horizontal tangent
  - $\frac{dx}{dt} = 0$ and $\frac{dy}{dt} = 0 \iff$ The particle is not moving

Name________________________________

Seat # ________ Date ____________________

Motion Problems in Parametric Equations

1. A particle's position at time t on the coordinate plane xy is given by the vector $\langle \sec t, \tan t \rangle$.
   a) Find the velocity vector at any time t. Use your expression to find the velocity vector for the particle at $t = \frac{\pi}{6}$.
   b) Find the acceleration vector at any time t. Use your expression to find the acceleration vector for the particle at $t = \frac{\pi}{6}$.
   c) Find the particle's speed at $t = \frac{\pi}{6}$.
   d) Find the equation of the line tangent to the motion of the particle at $t = \frac{\pi}{6}$.
   e) Set up an integral expression to find the total distance traveled by the particle in the time interval $\left[0, \frac{\pi}{3}\right]$. Use your calculator to evaluate your expression.

---

2. A particle's position at time t on the coordinate plane xy is given by the vector $\langle t^2 + t, 1 - t^3 \rangle$.
   a) Find the velocity vector of the particle at time $t = 4$.
   b) Find the acceleration vector of the particle at time $t = 4$.
   c) Find the particle's speed at time $t = 4$.
   d) Find the equation of the line tangent to the motion of the particle at $t = 2$.
   e) Set up an integral expression to find the total distance traveled by the particle in the time interval $[0, 4]$. Use your calculator to evaluate your expression.

(You may use your calculator to answer this question.)

3. The position of a particle moving on a plane is given by the parametric equations $x(t) = 3\sin t$ and $y(t) = t^2 - \cos t$, for $0 \le t \le \pi$.
   a) Find an equation for the normal line to the path described by the particle at $t = \frac{\pi}{3}$. The normal line is perpendicular to the tangent line to the curve.
   b) Find the initial and final position of the particle for $0 \le t \le \pi$. Find the distance between these two points.
   c) Find the total distance traveled by the particle for $0 \le t \le \pi$.
   d) Are your answers to parts (b) and (c) equal? Explain.

True or False? Explain…

4. The acceleration of an object is the derivative of its speed.

5. The velocity vector points in the direction of motion.

SEE OTHER SIDE

6. The velocity of a moving particle at time t is given by the vector $\langle t^3 - 4t,\ t \rangle$. At time $t = 1$, the position vector of the particle was $\left\langle 0,\ \frac{1}{2} \right\rangle$.

   a) Find the position vector of the particle at any time time t. To do so, solve the initial value problem $\frac{d\vec{r}}{dt} = \vec{v}$.
   b) Find the velocity vector at $t = 2$. Interpret the values found for both the horizontal and vertical velocity of the particle at time $t = 2$.
   c) Find the particle's speed at $t = 1$.
   d) Find the acceleration vector of the particle at any time time t.

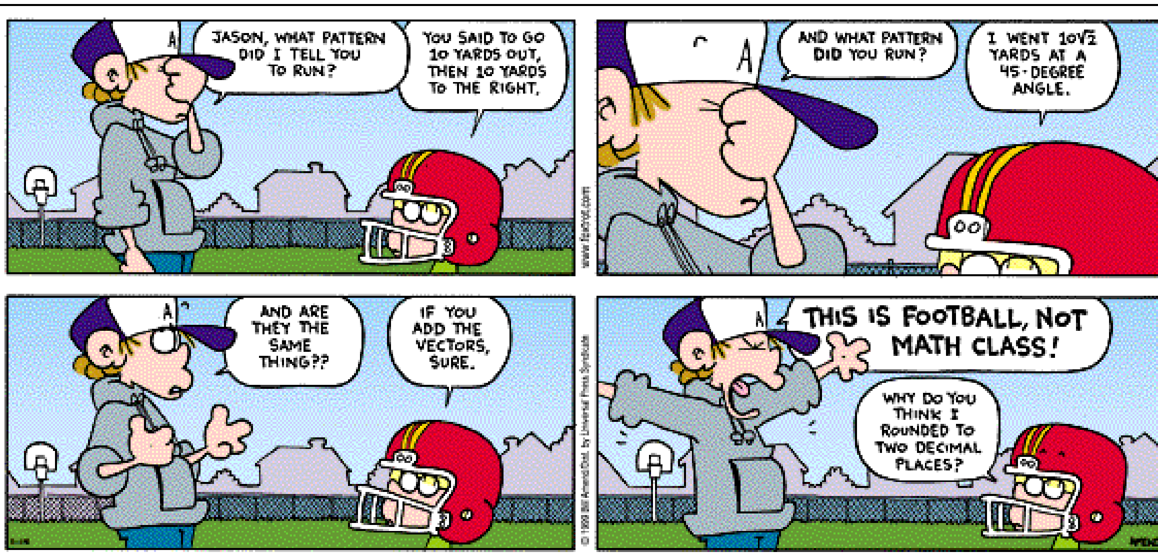
---

7. The position of a particle is given by the equations $x(t) = \sin t$ and $y(t) = \cos 2t$, for $0 \le t \le 2\pi$.
   a) Find the velocity vector for the particle at any time time t.
   b) For what value(s) of t is $\frac{d\vec{r}}{dt} = \vec{0}$?
   c) While describing the motion of the particle, what is the significance of the value(s) you found in part (b)? Explain.

***2000 CALCULUS BC** (no calculator allowed)*

8. A moving particle has position $(x(t), y(t))$ at time t. The position of the particle at time $t = 1$ is $(2, 6)$, and the velocity vector at any time $t > 0$ is given by $\left(1 - \frac{1}{t^2},\ 2 + \frac{1}{t^2} \right)$.

   a) Find the acceleration vector at time $t = 3$.
   b) Find the position of the particle at any time t. Use your formula to find the position of the particle at time $t = 3$.
   c) For what time $t > 0$ does the line tangent to the path of the particle at $(x(t), y(t))$ have a slope of 8?
   d) The particle approaches a line as $t \to \infty$. Find the slope of this line. Show the work that leads to your conclusion.



AP Calculus BC

CHAPTER 11 WORKSHEET

PARAMETRIC EQUATIONS AND POLAR COORDINATES

ANSWER KEY

1.

a) $\vec{v}=\langle \sec t\cdot\tan t,\ \sec^2 t\rangle$. At $t=\frac{\pi}{6}\Rightarrow \vec{v}=\left\langle \frac{2}{3},\ \frac{4}{3}\right\rangle$.

b) $\vec{a}=\langle \sec t\cdot\tan^2 t+\sec^3 t,\ 2\sec^2 t\tan t\rangle$. At $t=\frac{\pi}{6}\Rightarrow \vec{a}=\left\langle \frac{10\sqrt{3}}{9}, \frac{8\sqrt{3}}{9}\right\rangle\approx\langle 1.925,\ 1.540\rangle$.

c) Speed: $\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}$; at $t=\frac{\pi}{6}\Rightarrow\frac{2\sqrt{5}}{3}\approx 1.491$.

d) At $t=\frac{\pi}{6}\Rightarrow\frac{dy}{dx}=\frac{\frac{dy}{dt}}{\frac{dx}{dt}}=\frac{\frac{4}{3}}{\frac{2}{3}}=2$. Point: $\left(\frac{2\sqrt{3}}{3},\ \frac{\sqrt{3}}{3}\right)\Rightarrow y-\frac{\sqrt{3}}{3}=2\left(x-\frac{2\sqrt{3}}{3}\right)$

e) $\int_0^{\frac{\pi}{3}}\sqrt{(\sec t\cdot\tan t)^2+(\sec^2 t)^2}\,dt\approx 2.038$.

2.

a) $\vec{v}=\langle 9,\ -48\rangle$

b) $\vec{a}=\langle 2,\ -24\rangle$

c) Speed: $\sqrt{(9)^2+(-48)^2}=\sqrt{2385}\approx 48.837$

d) $t=2\Rightarrow\frac{dy}{dx}=\frac{\frac{dy}{dt}}{\frac{dx}{dt}}=\frac{-12}{5}$. Point: $(6,\ -7)\Rightarrow y+7=-\frac{12}{5}(x-6)$

e) $\int_0^4\sqrt{(2t+1)^2+(-3t^2)^2}\,dt\approx 68.209$

3.

a) $t=\frac{\pi}{3}\Rightarrow\frac{dy}{dx}=\frac{\frac{dy}{dt}}{\frac{dx}{dt}}=\frac{2t+\sin t}{3\cos t}\approx 1.974\Rightarrow$ Slope normal line: $-\frac{1}{1.974}\approx -0.507$.

Point: $(2.599,\ 0.597)\Rightarrow y-0.597=-0.507(x-2.599)$

b) Initial: $(0,\ -1)$. Final: $(0,\ \pi^2+1)$. Distance: $\pi^2+2\approx 11.870$.

c) $\int_0^{\pi}\sqrt{(3\cos t)^2+(2t+\sin t)^2}\,dt\approx 14.135$

d) No. One is the arc length the other one is a chord (line segment).

---

4. False. The acceleration is the derivative of the velocity.

---

5. True.

6.

a) Doing antiderivatives: $\vec{r} = \left\langle \frac{t^4}{4} - 2t^2 + C_1,\ \frac{t^2}{2} + C_2 \right\rangle$. Using the initial condition:

$\vec{r} = \left\langle \frac{t^4}{4} - 2t^2 + \frac{7}{4},\ \frac{t^2}{2} \right\rangle$

b) $\vec{v} = \langle 0,\ 2 \rangle$. The particle horizontal velocity is zero: it is not moving horizontally. The particle vertical velocity is 2: it is moving upwards.

c) Speed: $\sqrt{(-3)^2 + (1)^2} = \sqrt{10}$

d) $\vec{a} = \langle 3t^2 - 4,\ 1 \rangle$

7.

a) $\vec{v} = \langle \cos t,\ -2\sin 2t \rangle$

b) $\frac{d\vec{r}}{dt} = 0 \Rightarrow \cos t = 0$ and $-2\sin 2t = 0$. That happens when $t = \frac{\pi}{2}$ and $\frac{3\pi}{2}$.

c) Those are the times when the particle's velocity is zero: the particle is stopped at those times.

2000 CALCULUS BC *(no calculator allowed)*

8.

a) $\vec{a} = \langle 2t^{-3},\ -2t^{-3} \rangle$. At $t = 3 \Rightarrow \vec{a} = \left\langle \frac{2}{27},\ -\frac{2}{27} \right\rangle$

b) Doing antiderivatives: $\vec{r} = \left\langle t + \frac{1}{t} + C_1,\ 2t - \frac{1}{t} + C_2 \right\rangle$. Using the initial condition:

$\vec{r} = \left\langle t + \frac{1}{t},\ 2t - \frac{1}{t} + 5 \right\rangle$. At $t = 3$: $\vec{r} = \left\langle \frac{10}{3},\ \frac{32}{3} \right\rangle$

c) $\left(1 - \frac{1}{t^2},\ 2 + \frac{1}{t^2}\right) = \left(\frac{dx}{dt}, \frac{dy}{dt}\right)$

Since $8 = \frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{2 + \frac{1}{t^2}}{1 - \frac{1}{t^2}} \Rightarrow t = \sqrt{\frac{3}{2}}$

d) $\lim_{t\to\infty}\left(\frac{dy}{dx}\right) = \lim_{t\to\infty}\left(\frac{2 + \frac{1}{t^2}}{1 - \frac{1}{t^2}}\right) = 2$. The slope of the line is 2.
