Lagrange Error

#1 and #2 are tricky b/c no interval is given (usually the interval is given)... the trick is that IF no interval is provided, use the interval from the center of the series (we usually call it "a") to the value at which the fxn is being evaluated...
So, in #1, we'll use [0, .3] since it's a maclaurin series (centered about x=0) and being evaluated at .3.

In #2, we'll use [0,1] b/c it's the maclaurin series for e^x (centered about x=0) and being evaluated at 1.

① $\cos(.3) \approx 1 - \frac{(.3)^2}{2!} + \frac{(.3)^4}{4!}$

* Another trick:
$\cos(x) \approx 1 - \frac{x^2}{2!} + \frac{x^4}{4!}$ is actually both the 4th and 5th degree polynomial (since the 5th term in cos(x) Mac series has coefficient = 0)

So... call $\cos x \approx 1 - \frac{x^2}{2!} + \frac{x^4}{4!} = P_5(x)$

not $P_4(x)$...

(this is a rare trap... you are NOT likely to see it again...)

① (continued) Lagrange Unit

$|R_5(x)| \leq \left|\frac{f^{(6)}(x)}{6!}(x-0)^6\right|$

$f(x) = \cos x$
⋮ b/c cos x cycles in 4's...
$f^{(4)} = \cos x$
$f^{(5)}(x) = -\sin x$
$f^{(6)}(x) = -\cos x$

$|R_5(x)| \leq \left|\frac{-\cos x}{6!}(x)^6\right|$

I forgot to mention that the $(n+1)^{th}$ term should be made ⊕ by Abs. val.

in $[0, .3]$

x^6 is biggest when x is biggest, so use $.3^6$

$\cos x$ is biggest at $x=0$ (b/c cosine is 1 at $x=0$ and then decreases until $x=\pi$) so use $x=0$ for $\cos x$

$|R_5(x)| \leq \left|\frac{-\cos 0}{6!}(.3)^6\right|$

$|R_5(x)| \leq |-1.012_{(3)} \times 10^{-6}|$

$\boxed{|R_5(x)| \leq +1.012_{(3)} \times 10^{-6}}$

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② $e \approx 1 + 1 + \frac{1}{2} + \frac{1}{6} + \frac{1}{24}$

is $e^x \approx 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!}$ evaluated at $x=1$

so we have the 4th degree Maclaurin for e^x..

$P_4(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!}$

$f(x) = e^x$
$f'(x) = e^x$
⋮
$f^{(5)}(x) = e^x$

$|R_4(x)| \leq \left|\frac{f^{(5)}(x)}{5!}(x-0)^5\right|$

$|R_4(x)| \leq \left|\frac{e^x}{5!}(x)^5\right|$ on $[0,1]$

e^x is largest at $x=1$

x^5 is largest at $x=1$

$|R_4(x)| \leq \left|\frac{e^1}{5!}(1)^5\right|$

$\boxed{|R_4(x)| \leq .022_{(3)}}$

Lagrange error unit

③ Ⓐ 4th degree taylor for $\ln x$ about $x=4$

$f(x)=\ln x \longrightarrow f(4)=\ln 4$

$f'(x)=\frac{1}{x}=x^{-1}=1/x \longrightarrow f'(4)=1/4$

$f''(x)=-x^{-2}=-1/x^2 \qquad f''(4)=-1/16$

$f'''(x)=2x^{-3}=2/x^3 \qquad f'''(4)=2/64=1/32$

$f^{(4)}(x)=-6x^{-4}=\frac{-6}{x^4} \qquad f^{(4)}(4)=\frac{-6}{4^4}$

$$P_4(x)=\ln 4+\frac{1}{4}(x-4)\frac{-\frac{1}{16}}{2!}(x-4)^2+\frac{\frac{1}{32}}{3!}(x-4)^3-\frac{\frac{6}{4^4}}{4!}(x-4)^4$$

$$\boxed{P_4(x)=\ln 4+\frac{1}{4}(x-4)-\frac{1}{32}(x-4)^2+\frac{1}{192}(x-4)^3-\frac{6}{4^4 4!}(x-4)^4}$$

Ⓑ Lagrange on $[4, 4.5]$

$|R_4(x)| \le \frac{f^{(5)}(x)}{5!}(x-4)^5$

biggest x gives biggest $(x-4)^5$

on $[4, 4.5]$

$f^{(5)}(x)=\frac{24}{x^5}$ ← smaller denom = bigger #

$|R_4(x)| \le \frac{f^{(5)}(4)}{5!}(4.5-4)^5$

$|R_4(x)| \le \frac{24/4^5}{5!}(.5)^5$

$|R_4(x)| \le 6.103_{(4)} \times 10^{-6}$

↑ oops... about $x=3$...
bah!! restart...

(4) $f(3)=1,\ f'(3)=3,\ f''(3)=7,\ f'''(3)=5$

(A)

$$P_3(x)=f(3)+f'(3)(x-3)+\frac{f''(3)}{2!}(x-3)^2+\frac{f'''(3)}{3!}(x-3)^3$$

$$\boxed{P_3(x)=1+3(x-3)+\frac{7}{2}(x-3)^2+\frac{5}{6}(x-3)^3}$$

(B) $f^{(4)}(x)\le 6$ $\quad [2.9, 3]$

value makes $(x-3)^4$ largest ↓

$$|R_3(x)|\le\frac{f^{(4)}(x)}{4!}(x-3)^4\le\frac{6}{4!}(2.9-3)^4$$

$$|R_3(x)|\le\frac{f^{(4)}(x)}{4!}(x-3)^4\le\frac{6(.1)^4}{4!}=2.5\times10^{-5}$$

$$\boxed{|R_3(x)|\le 2.5\times10^{-5}}$$

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(5) $P_3(x)=7-9(x-2)^2-3(x-2)^3$ $\quad\boxed{\text{about } x=2}$

given $|f^{(4)}(x)|\le 6$ justify why $f(0)$ is negative by using the lagrange error estimate for $[0,2]$

chose $x=0$ b/c it maximizes $(x-2)^4$

$$|R_3(x)|\le\frac{f^{(4)}(x)}{4!}(x-2)^4\le\frac{6(x-2)^4}{4!}$$

$$|R_3(x)|\le\frac{f^{(4)}(x)}{4!}(0-2)^4\le\frac{6(-2)^4}{4!}$$

$\boxed{|R_3(x)|\le 4}$ ← error

⑤ (CONTINUED)

$|R_3(x)| \le 4$

ESTIMATE FOR $f(0)$ BASED ON $P_3(0)$

we know

$$P_3(0) = 7 - 9(-2)^2 - 3(-2)^3$$
$$= 7 - 9(4) - 3(-8)$$
$$= 7 - 36 + 24 = 7 - 12 = -5$$

$P_3(0) = -5$ ... since by the lagrange error bound, we know that $P_3(0)$ must be accurate within $\underline{\underline{4}}$, we know that $f(0)$ is between $-5-4=-9$ and $-5+4=-1$

so $-9 < f(0) < -1$, which means $f(0)$ must be negative.

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⑥ $P_n(x)$ for $\cos x$ about $x=0$
(b/c 0 is the center of the interval from $[-\pi, \pi]$

use graph to find n such that

$|P_n(x) - \cos x| < .001$

so basically find n such that

$|R_n(x)| < .0001$ For all x IN $[-\pi, \pi]$

Lagrange UNIT

(6) $|P_n(x) - \cos x| < .001$ IN $[-\pi, \pi]$

$P_n(x) = \sum_{k=0}^{n} \frac{(-1)^n x^{2n}}{2n!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} \ldots$

so centered at $x=0$ (since 0 is the middle of $-\pi$ to π)

$|P_n(x) - \cos x| < .001$

means

$|\text{error}| < .001$

$|R_n(x)| < .001$

we know $R_n(x) = \left| \frac{f^{(n+1)}(x)}{(n+1)!} (x-0)^{n+1} \right|$

$f(x) = \cos x$

$f'(x) = -\sin x$

$f''(x) = -\cos x$

$|R_1(x)| \leq \left| \frac{f^{(2)}(x)}{2!} x^2 \right|$

$|R_1(x)| \leq \left| -\frac{\cos x}{2} (x^2) \right|$

$\leq \frac{1}{2}(\pi)^2 = 4.934_{(5)}$ which is NOT $< .001$

Skipped A bunch b/c we have A LONG way to go to get to less than .001

$|R_{10}(x)| \leq \left| \frac{f^{(11)}(x)}{11!} x^{11} \right| =$

irrelevant which trig fxn -- ON $[-\pi, \pi]$ IT will max out as 1

$f^{(8)}(x) = \cos x$

$f^{(9)}(x) = -\sin x$

$f^{(10)}(x) = -\cos x$

$f^{(11)}(x) = \sin x$

$|R_{10}(x)| \leq \frac{1 (\pi)^{11}}{11!} = .007$

STILL NOT small enough..

NOTE that $f^{(n+1)}(x)$ will always max out as 1 ON $[-\pi, \pi]$ ← b/c over a full period, All sines and cosines max out ~~to~~ 1...

so $R_n(x) \leq \frac{1 \cdot \pi^{n+1}}{(n+1)!}$ ← b/c x^{n+1} will always max out at $x = \pi$ ON $[-\pi, \pi]$

Lagrange UNIT

6 (continued) ...

So, we can use

$\frac{\pi^{n+1}}{(n+1)!}$ to find $R_n(x)$ ...

IN my calculator, I use "x" in place of "n" and graph

$R_n(x) = \frac{\pi^{n+1}}{(n+1)!}$ AS $\underbrace{Y_1 = \frac{\pi\text{^}(x+1)}{((x+1)!)}}$

this is called Discrete as it will only graph for integer values of $x \geq 0$

So you CAN'T see the graph... but you can TRACE specific points...

set your window so the Y's are TINY... ($Y_{MIN} = -.01$, $Y_{MAX} = .01$)

TRACE: $x = 2$ $\quad Y_1 = R_2(x) = 5.167$

$x = 3$ $\quad Y_1 = R_3(x) = 4.058$

⋮

skip down..

$x = 10$ $\quad Y_1 = R_{10}(x) = .007$

$x = 11$ $\quad Y_1 = R_{11}(x) = .002$ (actually .00192..)

$x = 12$ $\quad \boxed{Y_1 = R_{12}(x) = .0004...} < .001$

~~$x = 13$ $\quad Y_1 = R_{13}(x) =$~~

So $\boxed{P_{12}(x)}$ is accurate within .001

Lagrange UNIT

7 $P_6(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} + o(x^6)$

IT's that SAME TRICK from #1...

A

$|R_6(x)| \leq 5\times10^{-4}$

ON an interval centered around $x=0$

$\left|\frac{f^{(7)}(x)}{7!}(x-0)^7\right| \leq 5\times10^{-4}$

$f(x) = \sin x$

$f^{(4)}(x) = \sin x$

$f^{(5)}(x) = \cos x$

$f^{(6)}(x) = -\sin x$

$f^{(7)}(x) = -\cos x$

$\left|\frac{f^{(7)}(x)(x^7)}{7!}\right| \leq 5\times10^{-4}$

$\left|\frac{x^7}{7!}\right| \leq 5\times10^{-4}$

*since interval will include $x=0$ (IT's center) $\cos x$ will max out as $\cos 0 = 1$

$|x^7| \leq 7!(5\times10^{-4})$

$|x^7| \leq 2.52$

IF $x^7 = 2.52$

$x = (2.52)^\wedge(1/7) = 1.141$

so

$|x| \leq 1.141$

$\boxed{\text{Interval is } (-1.141,\ 1.141) \text{ or } -1.141 < x < 1.141}$

B $\left|\sin x - \left(x - \frac{x^3}{3!} + \frac{x^5}{5!}\right)\right| \leq .0005$

easiest WAY is

$Y_1 = abs(\sin(x) - (x - x^3/3! + x^5/5!))$

$Y_2 = .0005$ $\boxed{\text{2ND}}$ $\boxed{\text{CALC}}$ $\boxed{\text{INT}}$

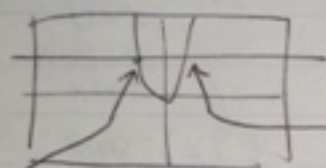
FLIP page

Lagrange error

7 (cont.)

$Y_1 = \text{Abs}(\sin(x) - (x - (x^\wedge 3)/6 + (x^\wedge 5)/120))$

$Y_2 = .0005$ ← so set y-max really small...
like $y_{max} = .001$
$y_{min} = -.001$



$\boxed{\text{2ND}}$ $\boxed{\text{CALC}}$ $\boxed{\text{INT}}$

$x \approx 1,144$

$x \approx -1,144$

$\boxed{\text{so interval is } (-1.144, 1.144) \text{ or } -1.144 < x < 1.144}$