Integration By Parts

Formula

$$\boxed{\int udv = uv - \int vdu}$$

I. Guidelines for Selecting u and dv:

(There are always exceptions, but these are generally helpful.)

"L-I-A-T-E" Choose 'u' to be the function that comes first in this list:

L: Logrithmic Function

I: Inverse Trig Function

A: Algebraic Function

T: Trig Function

E: Exponential Function

Example A: $\int x^3 \ln x \ dx$

*Since lnx is a logarithmic function and x^3 is an algebraic function, let:

$u = lnx$ (L comes before A in LIATE)

$dv = x^3\,dx$

$du = \dfrac{1}{x}\,dx$

$v = \int x^3 dx = \dfrac{x^4}{4}$

$$\begin{aligned}
\int x^3 \ln x dx &= uv - \int vdu \\
&= (\ln x)\frac{x^4}{4} - \int \frac{x^4}{4}\frac{1}{x}dx \\
&= (\ln x)\frac{x^4}{4} - \frac{1}{4}\int x^3 dx \\
&= \frac{x^4}{4}(\ln x) - \frac{1}{4}\,\frac{x^4}{4} + C \\
&= \frac{x^4}{4}(\ln x) - \frac{x^4}{16} + C \qquad \textbf{ANSWER}
\end{aligned}$$

Example B: $\int \sin x \ \ln(\cos x) \ dx$

u = ln(cosx) (Logarithmic Function)
dv *= sinx dx (Trig Function [L comes before T in LIATE])*

$$du = \frac{1}{\cos x}(-\sin x) \ dx = -\tan x \ dx$$

$$v = \int \sin x \ dx = -\cos x$$

$$\int \sin x \ \ln(\cos x) \ dx = uv - \int v du$$

$$= (\ln(\cos x))(-\cos x) - \int (-\cos x)(-\tan x) dx$$

$$= -\cos x \ \ln(\cos x) - \int (\cos x)\frac{\sin x}{\cos x} dx$$

$$= -\cos x \ \ln(\cos x) - \int \sin x \ dx$$

$$= -\cos x \ \ln(\cos x) + \cos x + C$$ **ANSWER**

Example C: $\int \sin^{-1} x \ dx$

*At first it appears that integration by parts does not apply, but let:

$u = \sin^{-1} x$ *(Inverse Trig Function)*

$dv = 1 \ dx$ *(Algebraic Function)*

$$du = \frac{1}{\sqrt{1-x^2}} \ dx$$

$$v = \int 1 dx = x$$

$$\int \sin^{-1} x \ dx = uv - \int v du$$

$$= (\sin^{-1} x)(x) - \int x \frac{1}{\sqrt{1-x^2}} \ dx$$

$$= x \ \sin^{-1} x - \left(-\frac{1}{2}\right)\int (1-x^2)^{-1/2}(-2x) \ dx$$

$$= x \ \sin^{-1} x + \frac{1}{2}(1-x^2)^{1/2}(2) + C$$

$$= x \ \sin^{-1} x + \sqrt{1-x^2} + C$$ **ANSWER**

II. Alternative General Guidelines for Choosing u and dv:

A. Let dv be the most complicated portion of the integrand that can be "easily' integrated.

B. Let u be that portion of the integrand whose derivative du is a "simpler" function than u itself.

Example: $\int x^3 \sqrt{4-x^2}\,dx$

*Since both of these are algebraic functions, the LIATE Rule of Thumb is not helpful. Applying Part (A) of the alternative guidelines above, we see that $x\sqrt{4-x^2}$ is the "most complicated part of the integrand that can easily be integrated." Therefore:

$$dv = x\sqrt{4-x^2}\,dx$$

$u = x^2$ (remaining factor in integrand)

$$du = 2x \quad dx$$

$$v = \int x\sqrt{4-x^2}\,dx = -\frac{1}{2}\int(-2x)(4-x^2)^{1/2} \quad dx$$

$$= \left(-\frac{1}{2}\right)\left(\frac{2}{3}\right)\ (4-x^2)^{3/2} = -\frac{1}{3}(4-x^2)^{3/2}$$

$$\int x^3\sqrt{4-x^3}\,dx = uv - \int vdu$$

$$= (x^2)\left(-\frac{1}{3}(4-x^2)^{3/2}\right) - \int -\frac{1}{3}(4-x^2)^{3/2}(2x) \quad dx$$

$$= \frac{-x^2}{3}(4-x^2)^{3/2} - \frac{1}{3}\int(4-x^2)^{3/2}(-2x) \quad dx$$

$$= \frac{-x^2}{3}(4-x^2)^{3/2} - \frac{1}{3}(4-x^2)^{5/2}\left(\frac{2}{5}\right) + C$$

$$= \frac{-x^2}{3}(4-x^2)^{3/2} - \frac{2}{15}(4-x^2)^{5/2} + C \quad \textbf{Answer}$$

III. Using repeated Applications of Integration by Parts:

Sometimes integration by parts must be repeated to obtain an answer.

Note: DO NOT switch choices for u and dv in successive applications.

Example: $\int x^2 \sin x \quad dx$

$u = x^2$ *(Algebraic Function)*

$dv = \sin x \quad dx$ *(Trig Function)*

$du = 2x \quad dx$

$v = \int \sin x \quad dx \quad = -\cos x$

$$\int x^2 \sin x \quad dx = uv - \int vdu$$

$$= x^2(-\cos x) - \int -\cos x \quad 2x \quad dx$$

$$= -x^2 \cos x + 2 \int x \quad \cos x \quad dx$$

Second application of integration by parts:

$u = x$ (Algebraic function) (Making "same" choices for u and dv)
$dv = \cos x$ (Trig function)
$du = dx$

$v = \int \cos x \quad dx \quad = \sin x$

$$\int x^2 \sin x \quad dx = -x^2 \cos x + 2 \quad [uv - \int vdu]$$

$$= -x^2 \cos x + 2 \quad [x \quad \sin x - \int \sin x \quad dx]$$

$$= -x^2 \cos x + 2 \quad [x \quad \sin x \quad + \quad \cos x \quad + \quad c]$$

$$= -x^2 \cos x \quad + \quad 2x \quad \sin x \quad + \quad 2\cos x \quad + \quad c$$ **Answer**

> Note: After each application of integration by parts, watch for the appearance of a constant multiple of the original integral.

Example: $\int e^x \cos x \quad dx$

$u = \cos x$ (Trig function)

$dv = e^x \quad dx$ (Exponential function)

$du = -\sin x \quad dx$

$v = \int e^x \quad dx \quad = e^x$

$$\int e^x \cos x \quad dx = uv - \int vdu$$

$$= \cos x \quad e^x - \int e^x (-\sin x) \quad dx$$

$$= \cos x \quad e^x + \int e^x \sin x \quad dx$$

Second application of integration by parts:

$u = \sin x$ (Trig function) (Making "same" choices for u and dv)

$dv = e^x \quad dx$ (Exponential function)

$du = \cos x \quad dx$

$v = \int e^x \quad dx \quad = e^x$

$$\int e^x \cos x \quad dx = e^x \cos x \quad + \quad (uv - \int vdu)$$

$$\int e^x \cos x \quad dx = e^x \cos x \quad + \quad \sin x \quad e^x - \int e^x \cos x \quad dx$$

Note appearance of original integral on right side of equation. Move to left side and solve for integral as follows:

$$2\int e^x \cos x \quad dx = \quad e^x \cos x \quad + \quad e^x \sin x \quad + \quad C$$

$$\int e^x \cos x \quad dx = \quad \frac{1}{2}(e^x \cos x \quad + \quad e^x \sin x) \quad + \quad C$$ **Answer**

Practice Problems:

1. $\int 3x \;\; e^{-x} \;\; dx$

2. $\int \frac{\ln x}{x^2} \;\; dx$

3. $\int x^2 \;\; \cos x \;\; dx$

4. $\int x \;\; \sin x \;\; \cos x \;\; dx$

5. $\int \cos^{-1} x \;\; dx$

6. $\int (\ln x)^2 \;\; dx$

7. $\int x^3 \;\; \sqrt{9 - x^2} \;\; dx$

8. $\int e^{2x} \;\; \sin x \;\; dx$

9. $\int x^2 \;\; \sqrt{x - 1} \;\; dx$

10. $\int \frac{1}{x(\ln x)^3} \;\; dx$

Solutions:

1. $-3xe^{-x} - 3e^{-x} + C$ $\quad u = 3x$, $dv = e^{-x}dx$

2. $-\frac{\ln x}{x} - \frac{1}{x} + C$ $\quad u = \ln x$, $dv = \frac{1}{x^2}dx$

3. $x^2 \sin x + 2x\cos x - 2\sin x + C$ $\quad u = x^2$, $dv = \cos x \quad dx$

4. $-\frac{x\cos 2x}{4} + \frac{\sin 2x}{8} + C$ $\quad$ note: $\frac{\sin 2x}{2} = \sin x \quad \cos x$, $u = x$, $dv = \sin 2x \quad \cos x \quad dx$

5. $x \quad \cos^{-1} x \quad - \quad \sqrt{1-x^2} + C$ $\quad u = \cos^{-1} x$, $dv = dx$

6. $x(\ln x)^2 - 2x \quad \ln x + 2x + C$ $\quad u = (\ln x)^2$, $dv = dx$

7. $-\frac{x^2}{3}(9-x^2)^{3/2} - \frac{2}{15}(9-x^2)^{5/2} + C$ $\quad u = x^2$, $dv = (4-x^2)^{1/2} \quad x \quad dx$

8. $\frac{2e^{2x}\sin x}{5} - \frac{e^{2x}\cos x}{5} + C$ $\quad u = \sin x$, $dv = e^{2x}dx$

9. $\frac{2x^2(x-1)^{3/2}}{3} - \frac{8x(x-1)^{5/2}}{15} + \frac{16(x-1)^{7/2}}{105} + C$ $\quad u = x^2$, $dv = (x-1)^{1/2}dx$

10. $\frac{-1}{2(\ln x)^2} + C$ $\quad u = \frac{1}{(\ln x)^3} = (\ln x)^{-3}$, $dv = \frac{1}{x}dx$