HA2PC S.Hogan
Intro to Calculus

Name: ________________________________ Packet Grade: _________/40

Schedule:

DAY	TOPIC	CW	HW
DAY 1	Limits graphically.	Example 1 (#1-21)	Exercises #1-29
DAY 2	Limits algebraically.	Examples #1-7	Supplemental Exercises #1-16
DAY 3	Difference quotients (The limit definition of the derivative)	Examples and #1-3	#4-8
DAY 4	Derivatives with the power rule.	#2-30 even	#1-29 odd
DAY 5	Writing the equations of tangent and normal lines.	#2-12 even	#1-13 odd

All CW and HW problems will be submitted on the day of the "Intro to Calc" Exam. You will need to do some problems on notebook paper because this packet does not have sufficient space for all work.

Section 2.1: Limits Graphically

Definition. We say that the **limit of $f(x)$ as x approaches a is equal to L**, written

$$\lim_{x\to a} f(x) = L,$$

if we can make the values of $f(x)$ as close to L as we like by taking x to be sufficiently close to a, *but not equal to a*. In other words, as x approaches a (but never equaling a), $f(x)$ approaches L.

Definition. We say that the **limit of $f(x)$ as x approaches a from the left is equal to L**, written

$$\lim_{x\to a^-} f(x) = L,$$

if we can make the values of $f(x)$ as close to L as we like by taking x to be sufficiently close to a, but *strictly less than a* (i.e., to the left of a as viewed on a number line). In other words, as x approaches a from the left (i.e., $x < a$), $f(x)$ approaches L.

Definition. We say that the **limit of $f(x)$ as x approaches a from the right is equal to L**, written

$$\lim_{x\to a^+} f(x) = L,$$

if we can make the values of $f(x)$ as close to L as we like by taking x to be sufficiently close to a, but *strictly greater than a* (i.e., to the right of a as viewed on a number line). In other words, as x approaches a from the right (i.e., $a < x$), $f(x)$ approaches L.

Definition. Limits taken from the left or the right are called **one-sided limits**.

Result. If both one-sided limits equal L, then the two-sided limit must also equal L. Conversely, if the two-sided limit equals L, then both one-sided limits must also equal L. That is,

$$\lim_{x\to a} f(x) = L \qquad \text{if and only if} \qquad \lim_{x\to a^-} f(x) = L \quad \text{and} \quad \lim_{x\to a^+} f(x) = L.$$

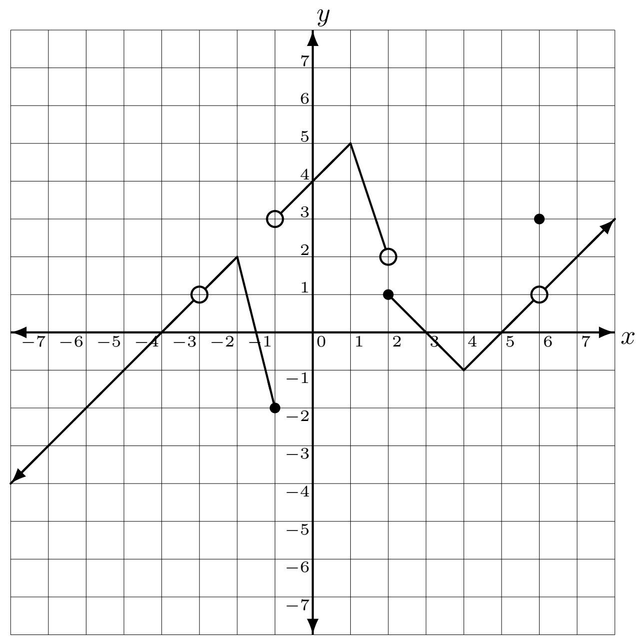
Definition. The function f is **continuous at** $x = a$ provided $f(a)$ is defined, $\lim_{x\to a} f(x)$ exists, and

$$\lim_{x\to a} f(x) = f(a).$$

In other words, the value of the limit equals the value of the function. Graphically, the function f is continuous at $x = a$ provided the graph of $y = f(x)$ does not have any holes, jumps, or breaks at $x = a$. (That is, the function is connected at $x = a$.)

If f is not continuous at $x = a$, then we say f is **discontinuous at** $x = a$ (or f has a **discontinuity at** $x = a$).

Example 1. For the function f graphed below, find the following:



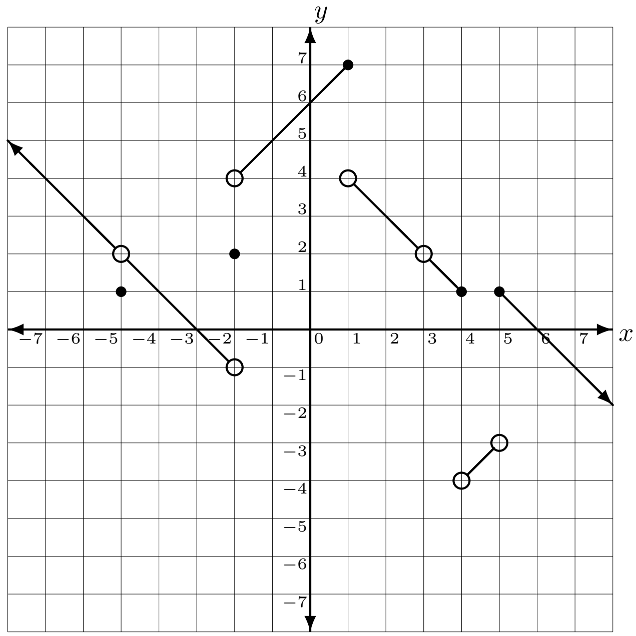
1. $\lim_{x\to -3^-} f(x) =$
2. $\lim_{x\to -3^+} f(x) =$
3. $\lim_{x\to -3} f(x) =$
4. $f(-3) =$
5. $\lim_{x\to -1^-} f(x) =$
6. $\lim_{x\to -1^+} f(x) =$
7. $\lim_{x\to -1} f(x) =$
8. $f(-1) =$
9. $\lim_{x\to 2^-} f(x) =$
10. $\lim_{x\to 2^+} f(x) =$
11. $\lim_{x\to 2} f(x) =$
12. $f(2) =$
13. $\lim_{x\to 4^-} f(x) =$
14. $\lim_{x\to 4^+} f(x) =$
15. $\lim_{x\to 4} f(x) =$
16. $f(4) =$
17. $\lim_{x\to 6^-} f(x) =$
18. $\lim_{x\to 6^+} f(x) =$
19. $\lim_{x\to 6} f(x) =$
20. $f(6) =$
21. List the value(s) of x at which f is discontinuous.

Note that the function is continuous at $x = 4$ and hence

$$\lim_{x\to 4} f(x) = \lim_{x\to 4^-} f(x) = \lim_{x\to 4^+} f(x) = f(4) = -1.$$

EXERCISES

For the function f graphed below, find the following:



1. $\lim_{x \to -5^-} f(x) =$
2. $\lim_{x \to -5^+} f(x) =$
3. $\lim_{x \to -5} f(x) =$
4. $f(-5) =$
5. $\lim_{x \to -2^-} f(x) =$
6. $\lim_{x \to -2^+} f(x) =$
7. $\lim_{x \to -2} f(x) =$
8. $f(-2) =$
9. $\lim_{x \to 0^-} f(x) =$
10. $\lim_{x \to 0^+} f(x) =$
11. $\lim_{x \to 0} f(x) =$
12. $f(0) =$
13. $\lim_{x \to 1^-} f(x) =$
14. $\lim_{x \to 1^+} f(x) =$
15. $\lim_{x \to 1} f(x) =$
16. $f(1) =$
17. $\lim_{x \to 3^-} f(x) =$
18. $\lim_{x \to 3^+} f(x) =$
19. $\lim_{x \to 3} f(x) =$
20. $f(3) =$
21. $\lim_{x \to 4^-} f(x) =$
22. $\lim_{x \to 4^+} f(x) =$
23. $\lim_{x \to 4} f(x) =$
24. $f(4) =$
25. $\lim_{x \to 5^-} f(x) =$
26. $\lim_{x \to 5^+} f(x) =$
27. $\lim_{x \to 5} f(x) =$
28. $f(5) =$
29. List the value(s) of x at which f is discontinuous.

ANSWERS

1. 2
2. 2
3. 2
4. 1
5. -1
6. 4
7. Does not exist
8. 2
9. 6
10. 6
11. 6
12. 6
13. 7
14. 4
15. Does not exist
16. 7
17. 2
18. 2
19. 2
20. Undefined
21. 1
22. -4
23. Does not exist
24. 1
25. -3
26. 1
27. Does not exist
28. 1
29. $x = -5, -2, 1, 3, 4, 5$

Section 2.1: Limits Algebraically

Recall. A function f is continuous at $x = a$ provided the graph of $y = f(x)$ does not have any holes, jumps, or breaks at $x = a$. (That is, the function is connected at $x = a$.)

If f is continuous at $x = a$, then

$$\lim_{x \to a} f(x) = f(a).$$

That is, the value of the limit equals the value of the function.

Result. Almost all of the functions you are familiar with are continuous at every number in their domain. In particular, the following functions (and any combinations of these functions) are continuous at every number in their domain:

- polynomials $\big($e.g., $f(x) = x^3 - 2x^2 + 7x - 2\big)$.
- rational functions $\left(\text{e.g.,}\quad f(x) = \dfrac{x^2 - 3x - 9}{x^2 - 2x - 3}\right)$
- radical functions $\big($e.g., $f(x) = \sqrt{2x - 5}\big)$
- exponential functions $\big($e.g., $f(x) = e^{3x}\big)$
- logarithmic functions $\big($e.g., $f(x) = \ln(3x - 8)\big)$

Hence, to find the limit of any of the above function as x approaches a, we simply evaluate that function at $x = a$.

Example 1. Find $\displaystyle\lim_{x \to 3} x^2 + 4x + 1.$

Solution. The function $f(x) = x^2 + 4x + 1$ is continuous at all values of x. (Just think of the graph of $y = x^2 + 4x + 1$: it is a parabola and there are no holes, breaks, or jumps in the graph. Hence, it is continuous at all values of x.) Therefore, to evaluate the limit, we simply evaluate the function:

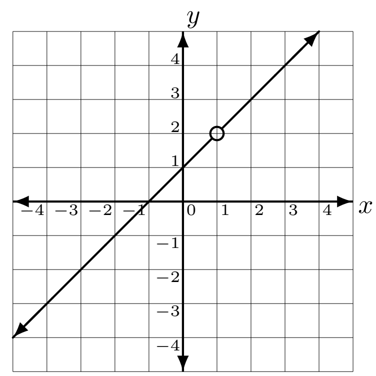
$$\lim_{x \to 3} x^2 + 4x + 1 = (3)^2 + 4(3) + 1 = \boxed{22}.$$

Example 2. Find $\displaystyle\lim_{x\to 1}\frac{x^2-1}{x-1}$.

SOLUTION. The function $f(x)=\frac{x^2-1}{x-1}$ is not continuous at $x=1$ since $f(1)=\left(\frac{0}{0}\right)$. Therefore, to find the limit, we must perform some algebra and eliminate the $\left(\frac{0}{0}\right)$ condition. In this case, we simplify the fraction:

$$\lim_{x\to 1}\frac{x^2-1}{x-1}=\lim_{x\to 1}\frac{\cancel{(x-1)}(x+1)}{\cancel{x-1}}=\lim_{x\to 1}x+1=(1)+1=\boxed{2}$$

The graph of $y=\frac{x^2-1}{x-1}$ is given to the right. Notice that the graph has a hole at $x=1$ and hence is not continuous there.



Example 3. Find $\displaystyle\lim_{x\to 2}\frac{\frac{x+2}{x}-2}{x-2}$.

SOLUTION. The function $f(x)=\frac{\frac{x+2}{x}-2}{x-2}$ is not continuous at $x=2$ since $f(2)=\left(\frac{0}{0}\right)$. Therefore, to find the limit, we must perform some algebra and eliminate the $\left(\frac{0}{0}\right)$ condition. In this case, we simplify the complex fraction:

$$\begin{aligned}\lim_{x\to 2}\frac{\frac{x+2}{x}-2}{x-2}=\lim_{x\to 2}\frac{\frac{x+2}{x}-\frac{2x}{x}}{x-2}=\lim_{x\to 2}\frac{\frac{x+2-2x}{x}}{x-2}&=\lim_{x\to 2}\frac{\frac{-x+2}{x}}{x-2}\\&=\lim_{x\to 2}\frac{-x+2}{x}\cdot\frac{1}{x-2}\\&=\lim_{x\to 2}\frac{-1\cancel{(x-2)}}{x}\cdot\frac{1}{\cancel{x-2}}\\&=\lim_{x\to 2}\frac{-1}{x}\\&=\boxed{-\frac{1}{2}}\end{aligned}$$

Graphically, $y=\dfrac{\frac{x+2}{x}-2}{x-2}$ will have a hole at $x=2$ and hence will not be continuous there.

Example 4. Find $\displaystyle\lim_{x\to 4} \frac{x^2-2x-8}{x-4} =$

Example 5. Find $\displaystyle\lim_{x\to -4} \frac{x^3+4x^2-x-4}{x^2+5x+4} =$

Example 6. Find $\displaystyle\lim_{x\to 0} \frac{\frac{1}{x+2} - \frac{1}{2}}{x} =$

Example 7. Find $\displaystyle\lim_{x\to 9} \frac{x-9}{\sqrt{x}-3} =$

Supplemental Exercises

Find the following limits:

1. $\lim_{x\to 3} x^2 + 2x - 7 =$

2. $\lim_{x\to 5} \dfrac{x^2 - 2x - 15}{x - 5} =$

3. $\lim_{x\to 1} \dfrac{4x^4 - 5x^2 + 1}{x^2 + 2x - 3} =$

4. $\lim_{x\to 2} \dfrac{(2x+1)^2 - 25}{x - 2} =$

5. $\lim_{x\to 1} \dfrac{\dfrac{2x}{x+1} - 1}{x - 1} =$

6. $\lim_{x\to -2} \dfrac{x^4 - 2x^2 - 8}{x^2 - x - 6} =$

7. $\lim_{x\to 0} \dfrac{x^2 + 7x + 6}{x + 3} =$

8. $\lim_{x\to 2} \dfrac{x^3 + x^2 - 4x - 4}{x^2 + x - 6} =$

9. $\lim_{x\to -1} \dfrac{\dfrac{1}{x} + 1}{x + 1} =$

10. $\lim_{x\to 0} \dfrac{(x+1)^2 - 1}{x} =$

11. $\lim_{x\to -3} \dfrac{2x^2 + 2x - 12}{x^2 + 4x + 3} =$

12. $\lim_{x\to 2} \dfrac{(3x-2)^2 - (x+2)^2}{x - 2} =$

13. $\lim_{x\to 2} \dfrac{\dfrac{2}{x^2} - \dfrac{1}{2}}{x - 2} =$

14. $\lim_{x\to 1} \dfrac{x - 1}{\sqrt{x} - 1} =$

15. $\lim_{x\to -2} \dfrac{\dfrac{x}{x+4} + 1}{x + 2} =$

16. $\lim_{x\to 3} \dfrac{x^2 - 2x - 3}{x + 5} =$

ANSWERS

1.	8	5.	1/2	9.	-1	13.	$-1/2$
2.	8	6.	24/5	10.	2	14.	2
3.	3/2	7.	2	11.	5	15.	1
4.	20	8.	12/5	12.	16	16.	0

Title: Finding Derivatives Using the Limit Definition
Class: Math 130 or Math 150
Author: Jason Miner
Instructions to tutor: Read instructions and follow all steps for each problem exactly as given.
Keywords/Tags: Calculus, derivative, difference quotient, limit

<u>Finding Derivatives Using the Limit Definition</u>

<u>Purpose:</u>
This is intended to strengthen your ability to find derivatives using the limit definition.

Recall that an expression of the form $\dfrac{f(x)-f(a)}{x-a}$ or $\dfrac{f(x+h)-f(x)}{h}$ is called a **difference quotient**.
For the definition of the derivative, we will focus mainly on the second of these two expressions. Before moving on to derivatives, let's get some practice working with the difference quotient.

The main difficulty is evaluating the expression $f(x+h)$, which seems to throw people off a bit.

Consider the function $f(x)=x^2-4x$. Let's evaluate this function at a few values.

$$f(2)=(2)^2-4(2)$$

$$f(0)=(0)^2-4(0)$$

$$f(-3)=(-3)^2-4(-3)$$

$$f(a)=(a)^2-4(a)$$

Note that we are just replacing the independent variable on each side of the equation with a particular value. So we should be able to do the same thing for $f(x+h)$: $f(x+h)=(x+h)^2-4(x+h)$

Now let's apply this to finding some difference quotients.

Example: Evaluate the difference quotient $\dfrac{f(x+h)-f(x)}{h}$ for the function $f(x)=x^2-4x$.

Now $\dfrac{f(x+h)-f(x)}{h}=\dfrac{[(x+h)^2-4(x+h)]-[x^2-4x]}{h}$.

Simplifying, $\dfrac{[(x+h)^2-4(x+h)]-[x^2-4x]}{h}=\dfrac{x^2+2xh+h^2-4x-4h-x^2+4x}{h}=\dfrac{2xh+h^2-4h}{h}$.

Note that we can reduce this fraction to obtain $\dfrac{2xh+h^2-4h}{h}=\dfrac{h(2x+h-4)}{h}=2x+h-4$.

Example: Now it's your turn. Evaluate the difference quotient for $f(x) = x^2 + 3x - 5$.

First find $f(x+h) = (\_\_\_\_\_\_)^2 + 3(\_\_\_\_\_\_) - 5$

Now $\dfrac{f(x+h) - f(x)}{h} = \dfrac{\left[(\_\_\_\_\_\_)^2 + 3(\_\_\_\_\_\_) - 5\right] - \left[(\_\_\_\_\_\_)^2 + 3(\_\_\_\_\_\_) - 5\right]}{h}$.

Did you get $\dfrac{[(x+h)^2 + 3(x+h) - 5] - [x^2 + 3x - 5]}{h}$? Good! Now simplify the numerator.

Did you get $\dfrac{2xh + h^2 + 3h}{h}$? If not, check that you distributed properly. Now factor the h out of the numerator and simplify the fraction.

You should have obtained $\dfrac{f(x+h) - f(x)}{h} = 2x + h + 3$. Try the next one on your own.

1. Evaluate the difference quotient $\dfrac{f(x+h) - f(x)}{h}$ for the function $f(x) = 2x^2 + 7x$.

Check your answer at the end of this document.

Now let's move on to finding derivatives. Recall the definition:

Limit-Definition of the Derivative: $$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

Example: Find the derivative of $f(x) = x^2 - 4x$.

In a previous example, we found $\dfrac{f(x+h) - f(x)}{h} = 2x + h - 4$.

So $f'(x) = \lim\limits_{h \to 0} \dfrac{f(x+h) - f(x)}{h} = \lim\limits_{h \to 0}(2x + h - 4) = 2x - 4$.

Example: Find the derivative of $f(x) = x^3 + 5x^2 - 4$.

Now $f'(x) = \lim\limits_{h \to 0} \dfrac{f(x+h) - f(x)}{h}$

$$= \lim_{h \to 0} \frac{[(x+h)^3 + 5(x+h)^2 - 4] - [x^3 + 5x^2 - 4]}{h}$$

$$= \lim_{h \to 0} \frac{(x^3 + 3xh^2 + 3x^2h + h^3) + (5x^2 + 10xh + 5h^2) - 4 - x^3 - 5x^2 + 4}{h}$$

$$= \lim_{h \to 0} \frac{3xh^2 + 3x^2h + h^3 + 10xh}{h}$$

$$= \lim_{h \to 0} \frac{h(3xh + 3x^2 + h^2 + 10x)}{h}$$

$$= \lim_{h \to 0}(3xh + 3x^2 + h^2 + 10x)$$

$$= 3x^2 + 10x$$

As you can see, the bulk of the work in finding the derivative is to evaluate and simplify the difference quotient.

You are on your own for the next two problems.

2. Find the derivative of each function using the limit definition.

   (a) $f(x) = x^2 + 3x - 5$ (Use your result from the first example on page 2 to help.)

   (b) $f(x) = 2x^2 + 7x$ (Use your result from the second example on page 2 to help.)

   (c) $f(x) = 4x^3 - 6x$ (Use the second example on page 3 as a guide.)

Check your answers at the end of this document.

Now you are ready to attempt these more challenging problems. You will need to employ the algebra skills you used in evaluating limits earlier, such as rationalizing techniques or adding rational expressions.

3. Find the derivative of each function using the limit definition.

(a) $f(x) = \sqrt{x}$

(b) $f(x) = \dfrac{1}{x}$

Check your answers – If you did not get these, consult a tutor for help.

1. $4x + 2h + 7$
2. (a) $f'(x) = 2x + 3$ (b) $f'(x) = 4x + 7$ (c) $f'(x) = 12x^2 - 6$
3. (a) $f'(x) = \dfrac{1}{2\sqrt{x}}$ (b) $f'(x) = -\dfrac{1}{x^2}$

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Day 3 Homework: Use the limit definition of the derivative to find f’x) for each function.

4. $f(x) = 5x + 2$

5. $f(x) = x^2 + 3x - 1$

6. $f(x) = 2x^3 + 4x + 7$

7. $f(x) = \sqrt{x + 3}$

8. $f(x) = \frac{4}{x^2}$

2.2 Worksheet #2 - More Power Rule Practice

Compute the derivatives of the following functions.

(1) $f(x) = x^2 - 2$	(2) $f(x) = x - x^3$	(3) $f(x) = x^2 + 3x - 6$
(4) $f(x) = 2x^2 - 4$	(5) $f(x) = \frac{2}{x}$	(6) $f(x) = \frac{4}{x^2} - \frac{x^2}{4}$
(7) $f(x) = 2x^{10} - 4x^2$	(8) $f(x) = 3\sqrt{x}$	(9) $f(x) = x\sqrt{3}$
(10) $f(x) = \frac{x^4}{4} + x - 2$	(11) $f(x) = x(x+1)$	(12) $f(x) = x^2 - e^2$
(13) $f(x) = 5x^3 - \frac{5}{x^3}$	(14) $f(x) = (6x+5) - (3x+x^2)$	(15) $f(x) = 2x^2 - 5x + 10$
(16) $f(x) = x - \frac{1}{x}$	(17) $f(x) = 4x^{\frac{5}{2}}$	(18) $f(x) = 1 - 5$
(19) $f(x) = \frac{1}{3x}$	(20) $f(x) = \frac{x^2}{2} - 3x$	(21) $f(x) = 5^2$
(22) $f(x) = (x^2+1)^2$	(23) $f(x) = x^{1000}$	(24) $f(x) = \frac{1}{x^{1000}}$
(25) $f(x) = \frac{x^2}{\ln(2)}$	(26) $f(x) = \sqrt{3x}$	(27) $f(x) = \sqrt{7}$
(28) $f(x) = \frac{x^2-1}{x}$	(29) $f(x) = \frac{8}{\sqrt{x}} - 3x$	(30) $f(x) = \frac{7x+3x^2}{5\sqrt{x}}$

Answers

(1)$2x$	(2)$1-3x^2$	(3)$2x+3$
(4)$4x$	(5)$-2x^{-2}$	(6)$-8x^{-3}-\frac{x}{2}$
(7)$20x^9-8x$	(8)$\frac{3}{2}x^{-\frac{1}{2}}$	(9)$\sqrt{3}$
(10)x^3+1	(11)$2x+1$	(12)$2x$
(13)$15x^2+15x^{-4}$	(14)$3-2x$	(15)$4x-5$
(16)$1+x^{-2}$	(17)$10x^{\frac{3}{2}}$	(18)0
(19)$-\frac{1}{3}x^{-2}$	(20)$x-3$	(21)0
(22)$4x^3+4x$	(23)$1000x^{999}$	(24)$-1000x^{-1001}$
(25)$\frac{2x}{\ln(2)}$	(26)$\frac{\sqrt{3}}{2\sqrt{x}}$	(27)0
(28)$1+x^{-2}$	(29)$-4x^{-\frac{3}{2}}-3$	(30)$\frac{7}{10}x^{-\frac{1}{2}}+\frac{9}{10}x^{\frac{1}{2}}$

AP Calculus AB - Worksheet 19 **Tangent and Normal Lines (Power Rule)**

Learn: Tangent and Normal Lines to a Curve
Recall: Derivative = slope of the Tangent line at that point's *x*-coordinate
Example:

$f(x) = x^2 + 3 \qquad (1,4)$

$f'(x) = 2x \Rightarrow f'(1) = 2 \rightarrow$ slope of the tangent line

Tangent Line: $y - 4 = 2(x-1)$

Normal Line: $y - 4 = -\frac{1}{2}(x-1)$

For each of the following:

a) Sketch a graph - USE GRAPH PAPER!!
b) Find the slope of the tangent line at the given point.
c) Find the equations of the tangent line at the given point. Sketch the line.
d) Find the equation of the normal to the curve at the given point. Sketch the line.

1. $y = x^2 - 3, \ (2,1)$
2. $f(x) = 6 - x^2 \quad (2,2)$
3. $f(x) = \sqrt{x}, \ (4,2)$
4. $y = 2 - 4x^{-2}, \quad (2,1)$

Find the equations of the tangent and normal lines to the curve at the given *x*-value.

5. $y = (1+2x)^2, \ x = 1$
6. $y = x^2(3-x), \ x = -2$
7. $y = x - \sqrt{x}, \ x = 4$

8. Find the points on the curve $y = 2x^3 + 3x^2 - 12x + 1$ where the tangent is horizontal.

9. For what values of *x* does the graph of $f(x) = (x^2+1)(x+3)$ have a horizontal tangent?

10. Show that the curve $y = 6x^3 + 5x - 3$ has no tangent line with slope 4.

11. Find an equation of the tangent line to the curve $y = x\sqrt{x}$ that is parallel to the line $y = 1 + 3x$.

12. Find equations of both lines that are tangent to the curve $y = 1 + x^3$ and are parallel to the line $12x - y = 1$.

13. Find a parabola with equation $y = ax^2 + bx + c$ that has slope 4 at $x = 1$, slope -8 at $x = -1$, and passes through the point $(2,15)$.

14. Evaluate $\lim_{\Delta x \to 0} \frac{(x+\Delta x)^3 - 2(x+\Delta x) - (x^3 - 2x)}{\Delta x}$.