

# SERIES CONVERGENCE/DIVERGENCE FLOW CHART

## TEST FOR DIVERGENCE

Does  $\lim_{n \rightarrow \infty} a_n = 0$ ?

NO

$\sum a_n$  Diverges

YES

## p-SERIES

Does  $a_n = 1/n^p, n \geq 1$ ?

YES

Is  $p > 1$ ?

YES

$\sum a_n$  Converges

NO

$\sum a_n$  Diverges

NO

## GEOMETRIC SERIES

Does  $a_n = ar^{n-1}, n \geq 1$ ?

YES

Is  $|r| < 1$ ?

YES

$\sum_{n=1}^{\infty} a_n = \frac{a}{1-r}$

NO

$\sum a_n$  Diverges

NO

## ALTERNATING SERIES

Does  $a_n = (-1)^n b_n$  or  $a_n = (-1)^{n-1} b_n, b_n \geq 0$ ?

YES

Is  $b_{n+1} \leq b_n$  &  $\lim_{n \rightarrow \infty} b_n = 0$ ?

YES

$\sum a_n$  Converges

NO

## TELESCOPING SERIES

Do subsequent terms cancel out previous terms in the sum? May have to use partial fractions, properties of logarithms, etc. to put into appropriate form.

YES

Does  $\lim_{n \rightarrow \infty} s_n = s$  finite?

YES

$\sum a_n = s$

NO

$\sum a_n$  Diverges

NO

## TAYLOR SERIES

Does  $a_n = \frac{f^{(n)}(a)}{n!} (x-a)^n$ ?

YES

Is  $x$  in interval of convergence?

YES

$\sum_{n=0}^{\infty} a_n = f(x)$

NO

$\sum a_n$  Diverges

NO

Try one or more of the following tests:

## COMPARISON TEST

Pick  $\{b_n\}$ . Does  $\sum b_n$  converge?

YES

Is  $0 \leq a_n \leq b_n$ ?

YES

$\sum a_n$  Converges

NO

Is  $0 \leq b_n \leq a_n$ ?

YES

$\sum a_n$  Diverges

NO

## LIMIT COMPARISON TEST

Pick  $\{b_n\}$ . Does  $\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = c > 0$  finite &  $a_n, b_n > 0$ ?

YES

Does  $\sum_{n=1}^{\infty} b_n$  converge?

YES

$\sum a_n$  Converges

NO

$\sum a_n$  Diverges

## INTEGRAL TEST

Does  $a_n = f(n), f(x)$  is continuous, positive & decreasing on  $[a, \infty)$ ?

YES

Does  $\int_a^{\infty} f(x) dx$  converge?

YES

$\sum_{n=a}^{\infty} a_n$  Converges

NO

$\sum a_n$  Diverges

## RATIO TEST

Is  $\lim_{n \rightarrow \infty} |a_{n+1}/a_n| \neq 1$ ?

YES

Is  $\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| < 1$ ?

YES

$\sum a_n$  Abs. Conv.

NO

$\sum a_n$  Diverges

## ROOT TEST

Is  $\lim_{n \rightarrow \infty} \sqrt[n]{|a_n|} \neq 1$ ?

YES

Is  $\lim_{n \rightarrow \infty} \sqrt[n]{|a_n|} < 1$ ?

YES

$\sum a_n$  Abs. Conv.

NO

$\sum a_n$  Diverges

1.  $\sum_{n=1}^{\infty} \frac{n^2 - 1}{n^2 + n}$
2.  $\sum_{n=1}^{\infty} \frac{n - 1}{n^2 + n}$
3.  $\sum_{n=1}^{\infty} \frac{1}{n^2 + n}$
4.  $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{n - 1}{n^2 + n}$
5.  $\sum_{n=1}^{\infty} \frac{(-3)^{n+1}}{2^{3n}}$
6.  $\sum_{n=1}^{\infty} \left( \frac{3n}{1 + 8n} \right)^n$
7.  $\sum_{n=2}^{\infty} \frac{1}{n\sqrt{\ln(n)}}$
8.  $\sum_{k=1}^{\infty} \frac{2^k k!}{(k + 2)!}$
9.  $\sum_{k=1}^{\infty} k^2 e^{-k}$
10.  $\sum_{n=1}^{\infty} n^2 e^{-n^3}$
11.  $\sum_{n=2}^{\infty} \frac{(-1)^{n+1}}{n \ln(n)}$
12.  $\sum_{n=1}^{\infty} (-1)^n \frac{n}{n^2 + 25}$
13.  $\sum_{n=1}^{\infty} \frac{3^n n^2}{n!}$
14.  $\sum_{n=1}^{\infty} \sin(n)$
15.  $\sum_{n=0}^{\infty} \frac{n!}{2 \cdot 5 \cdot 8 \cdots (3n + 2)}$
16.  $\sum_{n=1}^{\infty} \frac{n^2 + 1}{n^3 + 1}$
17.  $\sum_{n=1}^{\infty} (-1)^n 2^{1/n}$
18.  $\sum_{n=2}^{\infty} \frac{(-1)^{n-1}}{\sqrt{n} - 1}$
19.  $\sum_{n=1}^{\infty} (-1)^n \frac{\ln(n)}{\sqrt{n}}$
20.  $\sum_{k=1}^{\infty} \frac{k + 5}{5^k}$
21.  $\sum_{n=1}^{\infty} \frac{(-2)^{2n}}{n^n}$
22.  $\sum_{n=1}^{\infty} \frac{\sqrt{n^2 - 1}}{n^3 + 2n^2 + 5}$
23.  $\sum_{n=1}^{\infty} \tan(1/n)$
24.  $\sum_{n=1}^{\infty} \frac{\cos(n/2)}{n^2 + 4n}$
25.  $\sum_{n=1}^{\infty} \frac{n!}{e^{n^2}}$
26.  $\sum_{n=1}^{\infty} \frac{n^2 + 1}{5^n}$
27.  $\sum_{k=1}^{\infty} \frac{k \ln(k)}{(k + 1)^3}$
28.  $\sum_{n=1}^{\infty} \frac{e^{1/n}}{n^2}$
29.  $\sum_{n=1}^{\infty} \frac{\tan^{-1}(n)}{n\sqrt{n}}$
30.  $\sum_{j=1}^{\infty} (-1)^j \frac{\sqrt{j}}{j + 5}$
31.  $\sum_{k=1}^{\infty} \frac{5^k}{3^k + 4^k}$
32.  $\sum_{n=1}^{\infty} \frac{(2n)^n}{n^{2n}}$
33.  $\sum_{n=1}^{\infty} \frac{\sin(1/n)}{\sqrt{n}}$
34.  $\sum_{n=1}^{\infty} \frac{1}{n + n \cos^2(n)}$
35.  $\sum_{n=1}^{\infty} \left( \frac{n}{n + 1} \right)^{n^2}$
36.  $\sum_{n=2}^{\infty} \frac{1}{(\ln(n))^{\ln(n)}}$
37.  $\sum_{n=1}^{\infty} (\sqrt[n]{2} - 1)^n$
38.  $\sum_{n=1}^{\infty} (\sqrt[n]{2} - 1)$



## Answers

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[S Click here for solutions.](#)

1. D    3. C    5. C    7. D    9. C    11. C    13. C  
15. C    17. D    19. C    21. C    23. D    25. C  
27. C    29. C    31. D    33. C

### Solutions: Strategy for Testing Series

1.  $\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} \frac{n^2 - 1}{n^2 + 1} = \lim_{n \rightarrow \infty} \frac{1 - 1/n^2}{1 + 1/n} = 1 \neq 0$ , so the series  $\sum_{n=1}^{\infty} \frac{n^2 - 1}{n^2 + 1}$  diverges by the Test for Divergence.
3.  $\frac{1}{n^2 + n} < \frac{1}{n^2}$  for all  $n \geq 1$ , so  $\sum_{n=1}^{\infty} \frac{1}{n^2 + n}$  converges by the Comparison Test with  $\sum_{n=1}^{\infty} \frac{1}{n^2}$ , a  $p$ -series that converges because  $p = 2 > 1$ .
5.  $\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{(-3)^{n+2}}{2^{3(n+1)}} \cdot \frac{2^{3n}}{(-3)^{n+1}} \right| = \lim_{n \rightarrow \infty} \left| \frac{-3 \cdot 2^{3n}}{2^{3n} \cdot 2^3} \right| = \lim_{n \rightarrow \infty} \frac{3}{8} = \frac{3}{8} < 1$ , so the series  $\sum_{n=1}^{\infty} \frac{(-3)^{n+1}}{2^{3n}}$  is absolutely convergent by the Ratio Test.
7. Let  $f(x) = \frac{1}{x\sqrt{\ln x}}$ . Then  $f$  is positive, continuous, and decreasing on  $[2, \infty)$ , so we can apply the Integral Test. Since  $\int \frac{1}{x\sqrt{\ln x}} dx \left[ \begin{array}{l} u = \ln x, \\ du = dx/x \end{array} \right] = \int u^{-1/2} du = 2u^{1/2} + C = 2\sqrt{\ln x} + C$ , we find  $\int_2^{\infty} \frac{dx}{x\sqrt{\ln x}} = \lim_{t \rightarrow \infty} \int_2^t \frac{dx}{x\sqrt{\ln x}} = \lim_{t \rightarrow \infty} [2\sqrt{\ln x}]_2^t = \lim_{t \rightarrow \infty} (2\sqrt{\ln t} - 2\sqrt{\ln 2}) = \infty$ . Since the integral diverges, the given series  $\sum_{n=2}^{\infty} \frac{1}{n\sqrt{\ln n}}$  diverges.
9.  $\sum_{k=1}^{\infty} k^2 e^{-k} = \sum_{k=1}^{\infty} \frac{k^2}{e^k}$ . Using the Ratio Test, we get  $\lim_{k \rightarrow \infty} \left| \frac{a_{k+1}}{a_k} \right| = \lim_{k \rightarrow \infty} \left| \frac{(k+1)^2}{e^{k+1}} \cdot \frac{e^k}{k^2} \right| = \lim_{k \rightarrow \infty} \left[ \left( \frac{k+1}{k} \right)^2 \cdot \frac{1}{e} \right] = 1^2 \cdot \frac{1}{e} = \frac{1}{e} < 1$ , so the series converges.
11.  $b_n = \frac{1}{n \ln n} > 0$  for  $n \geq 2$ ,  $\{b_n\}$  is decreasing, and  $\lim_{n \rightarrow \infty} b_n = 0$ , so the given series  $\sum_{n=2}^{\infty} \frac{(-1)^{n+1}}{n \ln n}$  converges by the Alternating Series Test.
13.  $\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{3^{n+1}(n+1)^2}{(n+1)!} \cdot \frac{n!}{3^n n^2} \right| = \lim_{n \rightarrow \infty} \left[ \frac{3(n+1)^2}{(n+1)n^2} \right] = 3 \lim_{n \rightarrow \infty} \frac{n+1}{n^2} = 0 < 1$ , so the series  $\sum_{n=1}^{\infty} \frac{3^n n^2}{n!}$  converges by the Ratio Test.
15.  $\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{(n+1)!}{2 \cdot 5 \cdot 8 \cdots (3n+2)[3(n+1)+2]} \cdot \frac{2 \cdot 5 \cdot 8 \cdots (3n+2)}{n!} \right|$   
 $= \lim_{n \rightarrow \infty} \frac{n+1}{3n+5} = \frac{1}{3} < 1$   
 so the series  $\sum_{n=0}^{\infty} \frac{n!}{2 \cdot 5 \cdot 8 \cdots (3n+2)}$  converges by the Ratio Test.
17.  $\lim_{n \rightarrow \infty} 2^{1/n} = 2^0 = 1$ , so  $\lim_{n \rightarrow \infty} (-1)^n 2^{1/n}$  does not exist and the series  $\sum_{n=1}^{\infty} (-1)^n 2^{1/n}$  diverges by the Test for Divergence.

19. Let  $f(x) = \frac{\ln x}{x}$ . Then  $f'(x) = \frac{2 \cdot 5 \ln x}{2x^{3/2}} < 0$  when  $\ln x > 2$  or  $x > e^2$ , so  $\frac{\ln n}{n}$  is decreasing for  $n > e^2$ .  
By l'Hospital's Rule,  $\lim_{n \rightarrow \infty} \frac{\ln n}{n} = \lim_{n \rightarrow \infty} \frac{1/n}{1/(2 \cdot n)} = \lim_{n \rightarrow \infty} \frac{2}{n} = 0$ , so the series  $\sum_{n=1}^{\infty} (5 \cdot 1)^n \frac{\ln n}{n}$  converges by the Alternating Series Test.
21.  $\sum_{n=1}^{\infty} \frac{(5 \cdot 2)^{2n}}{n^n} = \sum_{n=1}^{\infty} \left(\frac{4}{n}\right)^n \cdot \lim_{n \rightarrow \infty} \sqrt[n]{|a_n|} = \lim_{n \rightarrow \infty} \frac{4}{n} = 0 < 1$ , so the given series is absolutely convergent by the Root Test.
23. Using the Limit Comparison Test with  $a_n = \tan\left(\frac{1}{n}\right)$  and  $b_n = \frac{1}{n}$ , we have  
 $\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = \lim_{n \rightarrow \infty} \frac{\tan(1/n)}{1/n} = \lim_{x \rightarrow \infty} \frac{\tan(1/x)}{1/x} \stackrel{H}{=} \lim_{x \rightarrow \infty} \frac{\sec^2(1/x) \cdot (5 \cdot 1/x^2)}{5 \cdot 1/x^2} = \lim_{x \rightarrow \infty} \sec^2(1/x) = 1^2 = 1 > 0$ .  
Since  $\sum_{n=1}^{\infty} b_n$  is the divergent harmonic series,  $\sum_{n=1}^{\infty} a_n$  is also divergent.
25. Use the Ratio Test.  $\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{(n+1)! \cdot e^{n^2}}{e^{(n+1)^2} \cdot n!} \right| = \lim_{n \rightarrow \infty} \frac{(n+1)n! \cdot e^{n^2}}{e^{n^2+2n+1}n!} = \lim_{n \rightarrow \infty} \frac{n+1}{e^{2n+1}} = 0 < 1$ , so  $\sum_{n=1}^{\infty} \frac{n!}{e^{n^2}}$  converges.
27.  $\int_2^{\infty} \frac{\ln x}{x^2} dx = \lim_{t \rightarrow \infty} \left[ \frac{\ln x}{x} - \frac{1}{x} \right]_1^t$  (using integration by parts)  $\stackrel{H}{=} 1$ . So  $\sum_{n=1}^{\infty} \frac{\ln n}{n^2}$  converges by the Integral Test, and since  $\frac{k \ln k}{(k+1)^3} < \frac{k \ln k}{k^3} = \frac{\ln k}{k^2}$ , the given series  $\sum_{k=1}^{\infty} \frac{k \ln k}{(k+1)^3}$  converges by the Comparison Test.
29.  $0 < \frac{\tan^{-1} n}{n^{3/2}} < \frac{\pi/2}{n^{3/2}} \cdot \sum_{n=1}^{\infty} \frac{\pi/2}{n^{3/2}} = \frac{\pi}{2} \sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$  which is a convergent  $p$ -series ( $p = \frac{3}{2} > 1$ ), so  $\sum_{n=1}^{\infty} \frac{\tan^{-1} n}{n^{3/2}}$  converges by the Comparison Test.
31.  $\lim_{k \rightarrow \infty} a_k = \lim_{k \rightarrow \infty} \frac{5^k}{3^k + 4^k} = [\text{divide by } 4^k] \lim_{k \rightarrow \infty} \frac{(5/4)^k}{(3/4)^k + 1} = )$  since  $\lim_{k \rightarrow \infty} \left(\frac{3}{4}\right)^k = 0$  and  $\lim_{k \rightarrow \infty} \left(\frac{5}{4}\right)^k = )$ .  
Thus,  $\sum_{k=1}^{\infty} \frac{5^k}{3^k + 4^k}$  diverges by the Test for Divergence.
33. Let  $a_n = \frac{\sin(1/n)}{n}$  and  $b_n = \frac{1}{n}$ . Then  $\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = \lim_{n \rightarrow \infty} \frac{\sin(1/n)}{1/n} = 1 > 0$ , so  $\sum_{n=1}^{\infty} \frac{\sin(1/n)}{n}$  converges by limit comparison with the convergent  $p$ -series  $\sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$  ( $p = 3/2 > 1$ ).