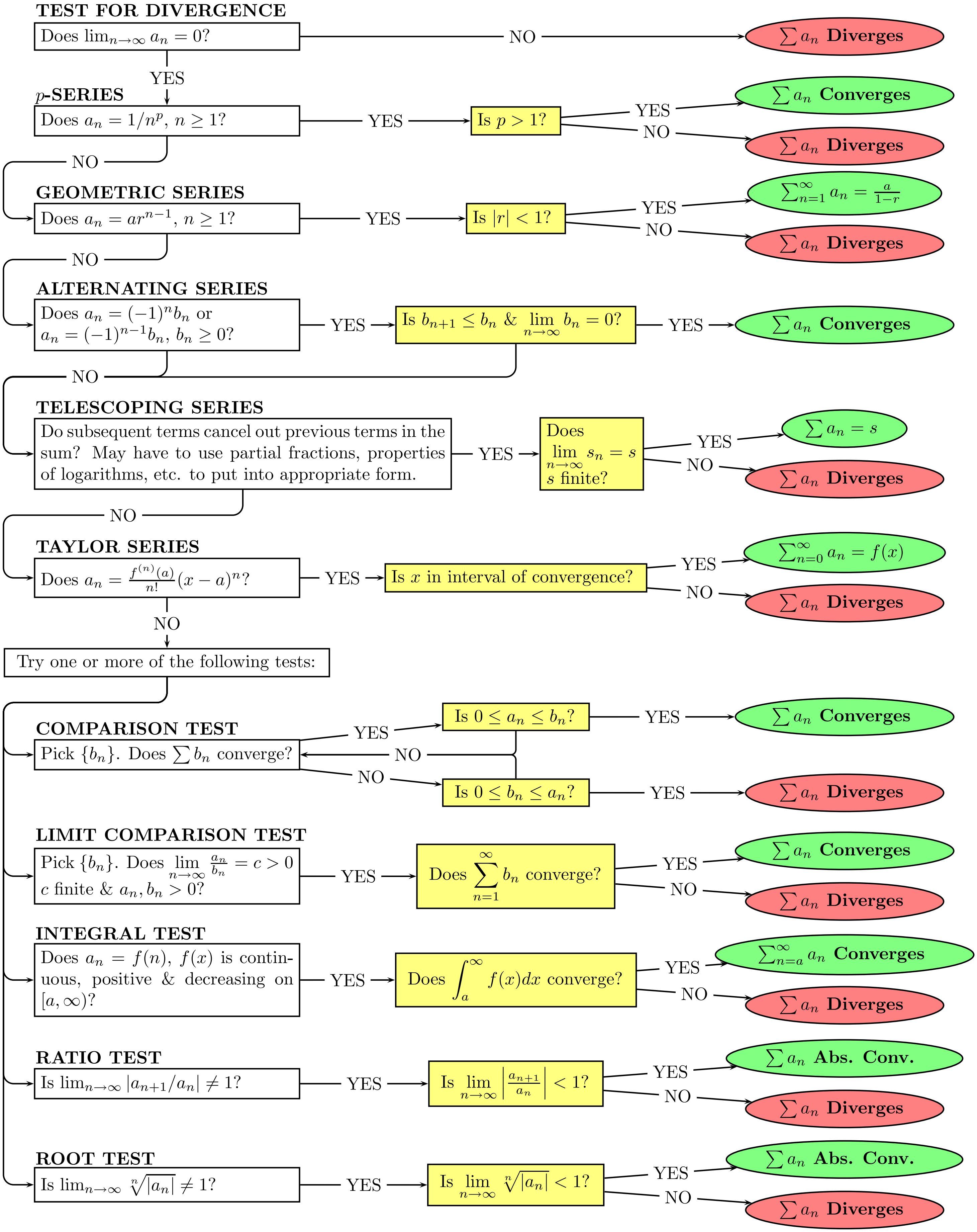
Series Convergence/Divergence Flow Chart



Problems 1-38 from Stewart's *Calculus*, page 784

1. $\sum_{n=1}^{\infty} \frac{n^2-1}{n^2+n}$

2. $\sum_{n=1}^{\infty} \frac{n-1}{n^2+n}$

3. $\sum_{n=1}^{\infty} \frac{1}{n^2+n}$

4. $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{n-1}{n^2+n}$

5. $\sum_{n=1}^{\infty} \frac{(-3)^{n+1}}{2^{3n}}$

6. $\sum_{n=1}^{\infty} \left(\frac{3n}{1+8n}\right)^n$

7. $\sum_{n=2}^{\infty} \frac{1}{n\sqrt{\ln(n)}}$

8. $\sum_{k=1}^{\infty} \frac{2^k k!}{(k+2)!}$

9. $\sum_{k=1}^{\infty} k^2 e^{-k}$

10. $\sum_{n=1}^{\infty} n^2 e^{-n^3}$

11. $\sum_{n=2}^{\infty} \frac{(-1)^{n+1}}{n\ln(n)}$

12. $\sum_{n=1}^{\infty} (-1)^n \frac{n}{n^2+25}$

13. $\sum_{n=1}^{\infty} \frac{3^n n^2}{n!}$

14. $\sum_{n=1}^{\infty} \sin(n)$

15. $\sum_{n=0}^{\infty} \frac{n!}{2 \cdot 5 \cdot 8 \cdot \dots \cdot (3n+2)}$

16. $\sum_{n=1}^{\infty} \frac{n^2+1}{n^3+1}$

17. $\sum_{n=1}^{\infty} (-1)^n 2^{1/n}$

18. $\sum_{n=2}^{\infty} \frac{(-1)^{n-1}}{\sqrt{n}-1}$

19. $\sum_{n=1}^{\infty} (-1)^n \frac{\ln(n)}{\sqrt{n}}$

20. $\sum_{k=1}^{\infty} \frac{k+5}{5^k}$

21. $\sum_{n=1}^{\infty} \frac{(-2)^{2n}}{n^n}$

22. $\sum_{n=1}^{\infty} \frac{\sqrt{n^2-1}}{n^3+2n^2+5}$

23. $\sum_{n=1}^{\infty} \tan(1/n)$

24. $\sum_{n=1}^{\infty} \frac{\cos(n/2)}{n^2+4n}$

25. $\sum_{n=1}^{\infty} \frac{n!}{e^{n^2}}$

26. $\sum_{n=1}^{\infty} \frac{n^2+1}{5^n}$

27. $\sum_{k=1}^{\infty} \frac{k\ln(k)}{(k+1)^3}$

28. $\sum_{n=1}^{\infty} \frac{e^{1/n}}{n^2}$

29. $\sum_{n=1}^{\infty} \frac{\tan^{-1}(n)}{n\sqrt{n}}$

30. $\sum_{j=1}^{\infty} (-1)^j \frac{\sqrt{j}}{j+5}$

31. $\sum_{k=1}^{\infty} \frac{5^k}{3^k+4^k}$

32. $\sum_{n=1}^{\infty} \frac{(2n)^n}{n^{2n}}$

33. $\sum_{n=1}^{\infty} \frac{\sin(1/n)}{\sqrt{n}}$

34. $\sum_{n=1}^{\infty} \frac{1}{n+n\cos^2(n)}$

35. $\sum_{n=1}^{\infty} \left(\frac{n}{n+1}\right)^{n^2}$

36. $\sum_{n=2}^{\infty} \frac{1}{(\ln(n))^{\ln(n)}}$

37. $\sum_{n=1}^{\infty} (\sqrt[n]{2}-1)^n$

38. $\sum_{n=1}^{\infty} (\sqrt[n]{2}-1)$

Answers

S Click here for solutions.

1. D **3.** C **5.** C **7.** D **9.** C **11.** C **13.** C
15. C **17.** D **19.** C **21.** C **23.** D **25.** C
27. C **29.** C **31.** D **33.** C

Solutions: Strategy for Testing Series

1. $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \dfrac{n^2-1}{n^2+1} = \lim_{n\to\infty} \dfrac{1-1/n^2}{1+1/n} = 1 \neq 0$, so the series $\sum_{n=1}^{\infty} \dfrac{n^2-1}{n^2+1}$ diverges by the Test for Divergence.

3. $\dfrac{1}{n^2+n} < \dfrac{1}{n^2}$ for all $n \geq 1$, so $\sum_{n=1}^{\infty} \dfrac{1}{n^2+n}$ converges by the Comparison Test with $\sum_{n=1}^{\infty} \dfrac{1}{n^2}$, a p-series that converges because $p = 2 > 1$.

5. $\lim_{n\to\infty} \left|\dfrac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\dfrac{(-3)^{n+2}}{2^{3(n+1)}} \cdot \dfrac{2^{3n}}{(-3)^{n+1}}\right| = \lim_{n\to\infty} \left|\dfrac{-3\cdot 2^{3n}}{2^{3n}\cdot 2^3}\right| = \lim_{n\to\infty} \dfrac{3}{2^3} = \dfrac{3}{8} < 1$, so the series $\sum_{n=1}^{\infty} \dfrac{(-3)^{n+1}}{2^{3n}}$ is absolutely convergent by the Ratio Test.

7. Let $f(x) = \dfrac{1}{x\sqrt{\ln x}}$. Then f is positive, continuous, and decreasing on $[2,\infty)$, so we can apply the Integral Test.
Since $\displaystyle\int \frac{1}{x\sqrt{\ln x}}\,dx \left[\begin{matrix} u = \ln x, \\ du = dx/x \end{matrix}\right] = \int u^{-1/2}\,du = 2u^{1/2} + C = 2\sqrt{\ln x} + C$, we find
$\displaystyle\int_2^{\infty} \frac{dx}{x\sqrt{\ln x}} = \lim_{t\to\infty} \int_2^t \frac{dx}{x\sqrt{\ln x}} = \lim_{t\to\infty} \left[2\sqrt{\ln x}\right]_2^t = \lim_{t\to\infty} \left(2\sqrt{\ln t} - 2\sqrt{\ln 2}\right) = \infty$. Since the integral diverges, the given series $\sum_{n=2}^{\infty} \dfrac{1}{n\sqrt{\ln n}}$ diverges.

9. $\sum_{k=1}^{\infty} k^2 e^{-k} = \sum_{k=1}^{\infty} \dfrac{k^2}{e^k}$. Using the Ratio Test, we get

$$\lim_{k\to\infty} \left|\frac{a_{k+1}}{a_k}\right| = \lim_{k\to\infty} \left|\frac{(k+1)^2}{e^{k+1}} \cdot \frac{e^k}{k^2}\right| = \lim_{k\to\infty} \left[\left(\frac{k+1}{k}\right)^2 \cdot \frac{1}{e}\right] = 1^2 \cdot \frac{1}{e} = \frac{1}{e} < 1,$$ so the series converges.

11. $b_n = \dfrac{1}{n\ln n} > 0$ for $n \geq 2$, $\{b_n\}$ is decreasing, and $\lim_{n\to\infty} b_n = 0$, so the given series $\sum_{n=2}^{\infty} \dfrac{(-1)^{n+1}}{n\ln n}$ converges by the Alternating Series Test.

13. $\lim_{n\to\infty} \left|\dfrac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\dfrac{3^{n+1}(n+1)^2}{(n+1)!} \cdot \dfrac{n!}{3^n n^2}\right| = \lim_{n\to\infty} \left[\dfrac{3(n+1)^2}{(n+1)n^2}\right] = 3\lim_{n\to\infty} \dfrac{n+1}{n^2} = 0 < 1$, so the series $\sum_{n=1}^{\infty} \dfrac{3^n n^2}{n!}$ converges by the Ratio Test.

15.
$$\begin{aligned}\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty} \left|\frac{(n+1)!}{2\cdot 5\cdot 8\cdot\cdots\cdot(3n+2)[3(n+1)+2]} \cdot \frac{2\cdot 5\cdot 8\cdot\cdots\cdot(3n+2)}{n!}\right| \\ &= \lim_{n\to\infty} \frac{n+1}{3n+5} = \frac{1}{3} < 1\end{aligned}$$

so the series $\sum_{n=0}^{\infty} \dfrac{n!}{2\cdot 5\cdot 8\cdot\cdots\cdot(3n+2)}$ converges by the Ratio Test.

17. $\lim_{n\to\infty} 2^{1/n} = 2^0 = 1$, so $\lim_{n\to\infty} (-1)^n 2^{1/n}$ does not exist and the series $\sum_{n=1}^{\infty} (-1)^n 2^{1/n}$ diverges by the Test for Divergence.

19. Let $f(x) = \frac{\ln x}{\overline{x}}$. Then $f'(x) = \frac{2\ 5\ \ln x}{2x^{3/2}} < 0$ when $\ln x > 2$ or $x > e^2$, so $\frac{\ln n}{\overline{n}}$ is decreasing for $n > e^2$.

By l'Hospital's Rule, $\lim_{n\to\infty} \frac{\ln n}{\overline{n}} = \lim_{n\to\infty} \frac{1/n}{1/\,(2\ \overline{n})} = \lim_{n\to\infty} \frac{2}{\overline{n}} = 0$, so the series $\sum_{n=1}^{\infty} (5\ 1)^n \frac{\ln n}{\overline{n}}$ converges by the Alternating Series Test.

21. $\sum_{n=1}^{\infty} \frac{(5\ 2)^{2n}}{n^n} = \sum_{n=1}^{\infty} \left(\frac{4}{n}\right)^n$. $\lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} \frac{4}{n} = 0 < 1$, so the given series is absolutely convergent by the Root Test.

23. Using the Limit Comparison Test with $a_n = \tan\left(\frac{1}{n}\right)$ and $b_n = \frac{1}{n}$, we have

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\tan(1/n)}{1/n} = \lim_{x\to\infty} \frac{\tan(1/x)}{1/x} \overset{\text{H}}{=} \lim_{x\to\infty} \frac{\sec^2(1/x)\cdot(5\ 1/x^2)}{5\ 1/x^2} = \lim_{x\to\infty} \sec^2(1/x) = 1^2 = 1 > 0.$$

Since $\sum_{n=1}^{\infty} b_n$ is the divergent harmonic series, $\sum_{n=1}^{\infty} a_n$ is also divergent.

25. Use the Ratio Test. $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{(n+1)!}{e^{(n+1)^2}} \cdot \frac{e^{n^2}}{n!}\right| = \lim_{n\to\infty} \frac{(n+1)n! \cdot e^{n^2}}{e^{n^2+2n+1} n!} = \lim_{n\to\infty} \frac{n+1}{e^{2n+1}} = 0 < 1$, so $\sum_{n=1}^{\infty} \frac{n!}{e^{n^2}}$ converges.

27. $\int_2^{\infty} \frac{\ln x}{x^2}\,dx = \lim_{t\to\infty} \left[\ \ \frac{\ln x}{x}\ \ \ \frac{1}{x}\right]_1^t$ (using integration by parts) $\overset{\text{H}}{=} 1$. So $\sum_{n=1}^{\infty} \frac{\ln n}{n^2}$ converges by the Integral Test, and since $\frac{k \ln k}{(k+1)^3} < \frac{k \ln k}{k^3} = \frac{\ln k}{k^2}$, the given series $\sum_{k=1}^{\infty} \frac{k \ln k}{(k+1)^3}$ converges by the Comparison Test.

29. $0 < \frac{\tan^{-1} n}{n^{3/2}} < \frac{\pi/2}{n^{3/2}}$. $\sum_{n=1}^{\infty} \frac{\pi/2}{n^{3/2}} = \frac{\pi}{2} \sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$ which is a convergent p-series ($p = \frac{3}{2} > 1$), so $\sum_{n=1}^{\infty} \frac{\tan^{-1} n}{n^{3/2}}$ converges by the Comparison Test.

31. $\lim_{k\to\infty} a_k = \lim_{k\to\infty} \frac{5^k}{3^k + 4^k} =$ [divide by 4^k] $\lim_{k\to\infty} \frac{(5/4)^k}{(3/4)^k + 1} =)$ since $\lim_{k\to\infty} \left(\frac{3}{4}\right)^k = 0$ and $\lim_{k\to\infty} \left(\frac{5}{4}\right)^k =)$. Thus, $\sum_{k=1}^{\infty} \frac{5^k}{3^k + 4^k}$ diverges by the Test for Divergence.

33. Let $a_n = \frac{\sin(1/n)}{\overline{n}}$ and $b_n = \frac{1}{n\ \ \overline{n}}$. Then $\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\sin(1/n)}{1/n} = 1 > 0$, so $\sum_{n=1}^{\infty} \frac{\sin(1/n)}{\overline{n}}$ converges by limit comparison with the convergent p-series $\sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$ ($p = 3/2 > 1$).