PART I: INDEFINITE INTEGRALS (no bounds)

TOPIC A: Properties of indefinite integrals

1. $\int f(x) \pm g(x)dx = \int f(x)dx \pm \int g(x)dx$

2. $\int k\, f(x)dx = k \int f(x)dx$

TOPIC B: Antiderivatives

You need to MEMORIZE these antiderivatives. There is no wiggle room here. Make note cards, quiz yourself, whatever it takes…but you need to have these in you noggin PERFECTLY.

1. $\int u^n\, du = \frac{u^{n+1}}{n+1} + C$

2. $\int \sin u \; du = -\cos u + C$

3. $\int \cos u \; du = \sin u + C$

4. $\int sec^2 u\, du = \tan u + C$

5. $\int csc^2 u\, du = -\cot u + C$

6. $\int \sec u \tan u \; du = \sec u + C$

7. $\int \csc u \cot u \; du = -\csc u + C$

8. $\int \frac{1}{u} du = \int \frac{du}{u} = ln|u| + C$

9. $\int e^u du = e^u + C$

10. $\int a^u du = \frac{1}{lna} a^u + C$

Shortcut for sines and cosines…you can use the table to help you derive (move DOWN) and integrate (move UP) sines and cosines.

DERIVE		
↓	**sin (u)**	↑
↓	**cos (u)**	↑
↓	**-sin (u)**	↑
	-cos (u)	**INTEGRATE**

NOTE: The purpose of u-substitution is to make a messy, complicated integral look like one of the ten antiderivatives listed above.

PRACTICE SECTION 1: Basic Integration

1. $\int 5\cos x\, dx =$

2. $\int 3csc^2 x\ dx =$

3. $\int 18sec^2 x\ dx =$

4. $\int -10\sec x \tan x\, dx =$

5. $\int 7\csc x \cot x\, dx =$

6. $\int 6x^2 dx =$

7. $\int 4x^5 + 3x - 8\ dx =$

8. $\int \frac{-3}{x} dx =$

9. $\int -9e^x dx =$

10. $\int -2 \cdot 4^x dx =$

TOPIC C: Simplifying before integrating.

After you clean these up, they will work as simple integration (using the antiderivatives that you totally have to memorize. Have I mentioned that those need to be MEMORIZED? Because they do. Most def. For serious.)

PRACTICE SECTION 2:

1. $\int \frac{8}{x^4} dx =$

2. $\int 3\sqrt{x} dx =$

3. $\int \frac{2x^2+3x-5}{x} dx =$

4. $\int 3x^3(5x^2 + 6x - 1)dx =$

5. $\int 5 \sec x \ (\sec x + \tan x)dx =$

6. $\int \frac{x^3+3x-5}{\sqrt{x}} dx =$

7. $\int (3x + 1)(x - 2)dx =$

TOPIC D: U-Substitution

When you integrate a FUNCTION WITHIN A FUNCTION, you need to use u-substitution to make the complicated integral look like one of those lovely memorize antiderivatives.

NOTE: U-substitution is INCREDIBLY COMMON…very few integrals will ever be as simple as the straight forward antiderivatives…so you need to get comfy with using u-sub. (When you get to BC, you will learn many other techniques that you will have to use when u-sub doesn't work….but in AB, u-sub is almost always the answer.)

HINTS FOR PICKING YOUR U:

1. It will likely be the INSIDE function (often inside parentheses, inside a trig function, in an exponent, inside a root, or in the denominator)
2. A MULTIPLE of the "du" will generally appear elsewhere in the integrand (which is good for canceling "x stuff")

PRACTICE SECTION 3: Basic u-substitution

1. $\int \sin 2x\, dx$

2. $\int \cos 3x\, dx$

3. $\int \sec 5x \tan 5x\, dx$

4. $\int e^{x^3-6x+3}(x^2-2)dx$

5. $\int \sqrt[3]{3x-7}dx$

6. $\int \frac{5x}{e^{x^2}}\,dx$

7. $\int \frac{3x-12}{x^2-8x+1}\,dx$

8. $\int x^2\sin(x^3)\,dx$

9. $\int \sin^4 x \cos x\,dx$

10. $\int \cot^5 x \csc^2 x\,dx$

11. $\int 2^{3x}\,dx$

12. $\int 3x(4x^2-5)^6\mathrm{dx}$

13. $\int 4\tan x \, dx$

14. $\int \frac{5}{(2x-3)^2} dx$

Sometimes, when you use u-substitution, you are fairly certain you've picked the correct "U" but some of the "x stuff" doesn't cancel. Since we can't integrate when there are multiple variables, we need to find a separate way (using our equation for "U") to replace the remaining "x stuff."

PRACTICE SECTION 4: Slightly more complicated u-substitution

(1) $\int 2x\sqrt{x-3}dx$

(2) $\int \frac{3x-1}{(x+1)^2} dx$

(3) $\int x^3 (x^2-4)^8 dx$

PART II: DEFINITE INTEGRALS (with bounds)

TOPIC E: Properties of definite integrals on $[a, b]$ with $a \le c \le b$.

1. $\int_a^b f(x) \pm g(x)dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)dx$
2. $\int_a^b k\,f(x)\,dx = k\int_a^b f(x)\,dx$
3. $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$
4. $\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx$
5. $\int_a^a f(x)\,dx = 0$
6. $\int_{-a}^a f(x)\,dx = 2\int_0^a f(x)\,dx$ for EVEN FXN f(x).
7. $\int_{-a}^a f(x)\,dx = 0$ for ODD FXN f(x).

TOPIC F: The First Fundamental Theorem of Calculus (FTC).

Let F(x) be the antiderivative of f(x) (here the capitalization matters…FYI.)

$$\int_a^b f(x)dx = [F(x)]_a^b = F(b) - F(a)$$

Meaning that you:

(1) integrate the function (after you integrate, you stop writing the "$\int \quad dx$" as those are telling you to integrate)
(2) write the antiderivative in brackets with the bounds as super and sub scripts on the right side
(3) evaluate the antiderivative at the top bound and subtract the value of the antiderivative evaluated at the bottom bound.

This is ALWAYS how we evaluate a definite integral.

NOTE: Your calculator can evaluate DEFINITE integrals. You do this by using MATH 9 to get "FNINT("…the calculator syntax is as follows:

$$\int_a^b f(x)\,dx = FNINT(f(x), x, a, b)$$

If the function is really complicated, it might be worth putting the function into the Y_1 spot and then using FNINT(Y_1, X, A, B)

PRACTICE SECTION 5: The 1st FTC—Evaluate each integral

(by hand AND on your calculator to check your work)

(1) $\int_1^3 x^2\,dx$

(2) $\int_0^{\pi/4} \sin(3x)\,dx$

(3) $\int_0^4 e^{2x}\,dx$

(4) $\int_1^3 \frac{1}{x}\,dx$

(5) $\int_0^2 2x^3 - 4x + 3\,dx$

(6) $\int_0^4 \sqrt{2x+1}\,dx$

(7) $\int_{\pi/4}^{\pi/3} \sec^2 x\,dx$

TOPIC G: The Mean Value Theorem (MVT) for Integrals

For a continuous function on $[a, b]$ with $a \le c \le b$.

$$\int_a^b f(x)\,dx = (b-a)f(c)$$

For at least one c. f(c) is called the "Average value of f(x) on [a, b]"

TOPIC H: The Average Value of a Function

$$\frac{1}{(b-a)}\int_a^b f(x)\,dx = f(c)$$

I always remember this as the "integral divided by the width of the interval." This makes sense if you picture a rectangle…the area of a rectangle is width times height (A = W*H)

If you solve for height, you get H = A/W. Thus, the average height (y-value) would be the AREA (a.k.a. integral) divided by the WIDTH (a.k.a. "b minus a".)

PRACTICE SECTION 6: Find the average value of a function on a given interval.

(1) $f(x) = 3x^2 \; on \; [1, 4]$

(2) $f(x) = x^2 - 4x \; on \; [0, 5]$

(3) $f(x) = \frac{2}{x^2} \; on \; [1, 3]$

(4) $f(x) = \frac{5}{x} \; on \; [1, e]$

(5) $f(x) = \sin x \; on \; [0, {}^{\pi}/_{3}]$

(6) $f(x) = e^{3x} \; on \; [1, 5]$

TOPIC I: The Second Fundamental Theorem of Calculus (FTC)

$$\frac{d}{dx}\int_{g(x)}^{h(x)} f(t)dt = f\big(h(x)\big)\cdot h'(x) - f(g(x))\cdot g'(x)$$

NOTE: Most of the time, when you see this, one of the bounds is a constant, so it's derivative is zero. The keys to using the 2nd FTC are:

(1) the derivative and the integral "cancel"
(2) the bounds get plugged into the function that is in the integrand
(3) because of the chain rule, you have to derive the bounds after you plug them in

PRACTICE SECTION 7: 2nd FTC

Use the 2nd FTC to evaluate each derivative.

(1) $\frac{d}{dx}\int_6^{5x} \sin t\, dt$

(2) $\frac{d}{dx}\int_3^{x^3} e^t dt$

(3) $\frac{d}{dx}\int_{sinx}^{10} \sqrt{2t+3}dt$

(4) $\frac{d}{dx}\int_{6x}^{3x^2} \ln t\, dt$

TOPIC J: Integrals that are best evaluated by geometry

Because $\int_a^b f(x)\,dx$ means the AREA under the curve from [a,b], sometimes we can use simple geometric formulae as a faster, more efficient way to integrate. This is most common for:

(1) Linear absolute values (which graph like a V…so the area ends up being two triangles.) To figure out where the "point" of the V happens, set in inside of the linear absolute value equal to zero and solve for x. Sometimes, depending on the interval, you may also have a piece that is a trapezoid rather than a triangle.

(2) Semicircles, quarter circles, and circles (which generally have the equation $y = \pm\sqrt{r^2 - x^2}$ with radius = r, when centered at the origin

  a. So $y = \sqrt{r^2 - x^2}$ from [-r, r] is a semicircle of radius r. (the TOP half of the circle)

  b. So $y = -\sqrt{r^2 - x^2}$ from [-r, r] is a semicircle of radius r. (the BOTTOM half of the circle)

  c. Either of the two listed above would be a QUARTER circle if the interval was [0, r] or [-r, 0]

IN ALL OF THESE CASES, SKETCH A QUICK GRAPH….IT IS TOTALLY WORTH IT IN THESE SCENARIOS…I PROMISE.

NOTE: The area of a triangle is $A = \frac{1}{2}bh$, the area of a circle is $A = \pi r^2$, and the area of a trapezoid is $A = \frac{1}{2}h(b_1 + b_2)$…these are obviously three formulae that you should have memorized from geometry.

PRACTICE SECTION 8: Evaluate these integrals by sketching the graph and using geometry.

(1) $\int_{-5}^{5} \sqrt{25 - x^2}\,dx$

(2) $\int_{-3}^{3} 2\sqrt{9 - x^2}\,dx$

(3) $\int_{0}^{10} \sqrt{100 - x^2}\,dx$

(4) $\int_{-6}^{0} -3\sqrt{36 - x^2}\, dx$

(5) $\int_{0}^{5} |x - 3|\, dx$

(6) $\int_{0}^{6} |2x - 4|\, dx$

(7) $\int_{1}^{4} |2x - 3|\, dx$

(8) $\int_{1}^{3} |2x + 1|\, dx$

PART III: APPLICATIONS OF INTEGRALS

TOPIC K: Solving differential equations (Diff EQs)

When you solve diff EQs, you will be integrating WITHOUT BOUNDS. This means that every time you integrate, you will need to write a +C. Then you will use a give point (often called an "initial value") to find the value of C.

If you have to integrate multiple times, you need to find the "C" at every step before you do the next round of integrating.

(1) YOU MUST WRITE BOTH THE INTEGRAL SYMBOL AND THE D___ (insert correct variable of integration here.) You cannot have an integral symbol without having the "dx" (though it is not always an x.) If the variable of integration is missing, the integral will immediately lose credit on the AP.

(2) YOU MUST WRITE WHICH FUNCTION YOU HAVE IN EACH STEP (f(x), f'(x), f"(x))…and you can only say things are equal if there are actually equal. WRITE WITH PURPOSE.

PRACTICE SECTION 9: Solving Diff EQs

Given the information, write the equation for f(x).

(1) $f'(x) = 2x, f(1) = 10$

(2) $f'(x) = \sin x\,, f\left(\frac{\pi}{2}\right) = 2$

(3) $f'(x) = e^{3x}, f(0) = -4$

(4) $f''(x) = 6x, f'(-1) = 5, f(0) = 3$

(5) $f''(x) = 4x + 5, f'(2) = 5, f(0) = 2$

(6) $f''(x) = \sqrt{x}, f'(4) = 0, f(1) = 5$

TOPIC L: AVP (Acceleration, Velocity, and Position) Problems—often called particle motion problems.

These are basically the same as solving any other diff EQ, except the problems are put in context and often have units.

RELATIONSHIP TO KNOW:

(1) Let's call POSITION x(t) (though sometimes it is h(t) or y (t) for height)—it is measured in units of length (like meters)
(2) VELOCITY is the DERIVATIVE OF POSITION (v(t) = x'(t))—it is measured in units of length over time (like meters per second)
(3) ACCELERATION is the DERIVATIVE ON VELOCITY (a(t) = v'(t) = x"(t))—it is measured in units of length per time squared (like meters per second squared)
(4) SPEED is the absolute value of VELOCITY (thus SPEED = $|v(t)|$)

WORDS AND PHRASES TO KNOW:

(1) "At rest" means VELOCITY IS EQUAL TO ZERO
(2) "Initially" means TIME (T) IS EQUAL TO ZERO
(3) "Speeding up" means v(t) and a(t) have the same sign (both positive or both negative)
(4) "Slowing down" means v(t) and a(t) have opposite signs (one is positive and the other is negative)
(5) "Displacement" means the difference between the final and initial positions, it can be found in two ways (here, let's say on the interval [a, b])
    a. x(b) – x(a) = final position minus initial position = displacement
    b. By integrating the VELOCITY….$\int_a^b v(t)dt = \int_a^b x'^{(t)}dt = x(b) - x(a)$
       ** this is good if you don't have a point to find the +C in the x(t) equation
(6) "Total Distance" means how far the particle traveled in total (regardless of direction)…this can be found in three ways (here let's say on the interval [a, b])
    a. If you have a calculator, integrate the SPEED…so TOTAL DISTANCE = $\int_a^b |v(t)|dt$
    b. If you don't have a calculator, find all of the times when the particle stops (a.k.a. is AT REST)…then integrate between those stops, and absolute value each result. Let's say the particle stops at x = c and x = d, then TOTAL DISTANCE = $\left|\int_a^c v(t)dt\right| + \left|\int_c^d v(t)dt\right| + \left|\int_d^b v(t)dt\right|$
    c. Lastly, if you already have a formula for x(t), you can just find the position at each "stop" (when v(t) = 0) and then calculator how far the particle travels from stop to stop by subtracting the positions.

PRACTICE SECTION 10: AVP Problems

1. (Non-Calculator) The velocity of a moving particle on a coordinate line is given by $v(t) = t^2 + 3t - 10$ ft/min, where t is measured in minutes and $t \geq 0$.
   a. Find the displacement of the particle during the first 3 minutes.
   b. Find when the particle's speed is decreasing. Justify.
   c. Write, but do not evaluate, an integral expression to find the total distance traveled by the particle for the first 5 minutes.

2. (Non-Calculator) Given the acceleration of the particle is $a(t) = -4$ ft/sec^2 and $v(0) = 12$ ft/sec during the interval $0 \le t \le 8$.
   a. Find the average velocity of the particle for the interval $0 \le t \le 8$.
   b. Find when the instantaneous velocity of the particle is equal to the average velocity from part (a).
   c. Find when the velocity is increasing.
   d. Find the total distance traveled by the particle during the interval $0 \le t \le 8$.

3. (Calculator) The velocity function of a moving particle is $v(t) = 3\cos(2t)$ in/hr for $0 \le t \le 2\pi$ hours.
   a. Determine when the particle is moving to the right. Justify.
   b. Find the total distance traveled by the particle during the time interval $0 \le t \le 2\pi$ hours.
   c. Given $x(0) = 5$, find $x(6)$.
   d. Find when the particle is speeding up.

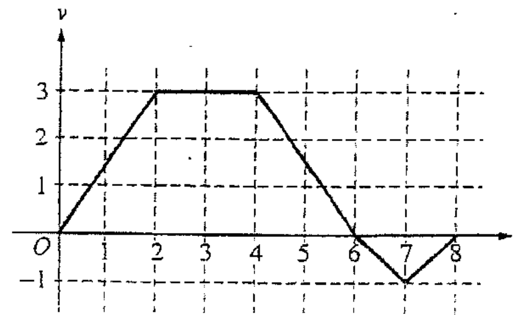
4. (Calculator) A particle moves along the x-axis so that its velocity at any time $t \geq 0$ is given by $v(t) = 3t^2 - 2t - 1$. The position $x(t)$ is 5 for $t = 2$.

   a) Write a polynomial expression for the position of the particle at any time $t \geq 0$.

   b) For what values of t, $0 \leq t \leq 3$, is the particle's instantaneous velocity the same as its average velocity on the closed interval $[0,3]$?

   c) Find the total distance traveled by the particle from time $t = 0$ until time $t = 3$.

Problems 5 and 6 use the graph shown below.



A bug begins to crawl up a vertical wire at time $t = 0$. The velocity v of the bug at time t, $0 \le t \le 8$, is given by the function whose graph is shown above.

5. (Non-Calculator) At what value of t does the bug change direction?

(A) 2 (B) 4 (C) 6 (D) 7 (E) 8

6. (Non-Calculator) What is the total distance the bug traveled from $t = 0$ to $t = 8$?

(A) 14 (B) 13 (C) 11 (D) 8 (E) 6