APAB - ch 6/7 review solns

PART I: Calculator permitted

FRQ (A) FIND **BOUNDS** $y_1 = \ln x$ $y_2 = x-2$

$\ln x = x - 2$

$$\boxed{\begin{array}{l} x = .15859434 = P \\ x = 3.1461932 = Q \\ \text{Area} = \int_P^Q \ln x - (x-2)\,dx \end{array}} \approx \boxed{1.949}$$

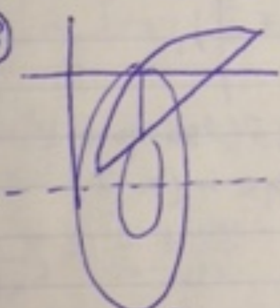
you need this to get full CREDIT

IN CALC, you'll type

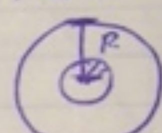
FNINT$(Y_1 - Y_2, X, P, Q)$

(this gets you no credit on AP)

(B)



IT'S A WASHER



$R = \ln x - (-3)$

$r = (x-2) - (-3)$

$$V = \pi \int_P^Q R^2 - r^2\,dx = \pi \int_P^Q (\ln x - (-3))^2 - (x - 2 - (-3))^2\,dx$$

TO GET CREDIT, you need to write:

$$V = \pi \int_P^Q (\ln x - (-3))^2 - (x - 2 - (-3))^2\,dx$$

*NOTE: you Don't have to clean this up AT ALL

CONTINUED ON NEXT PAGE

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FR5B IN CALC, you'll type

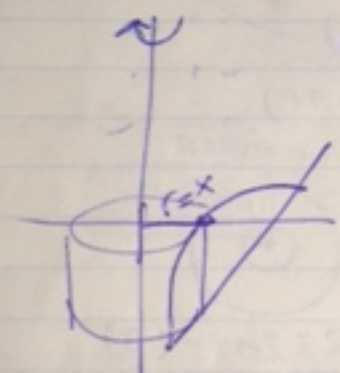
π FNINT $\left((Y_1 - (-3))^2 - (Y_2 - (-3))^2, X, P, Q \right)$

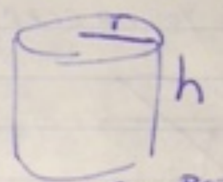
* but you'll get no credit for the calculator notation

$\boxed{\approx 34.198 \; (9)}$ ← AP credit for ANS.

(C) *WRITE but DO not evaluate*

IT's shells (b/c AOR || to "slice")



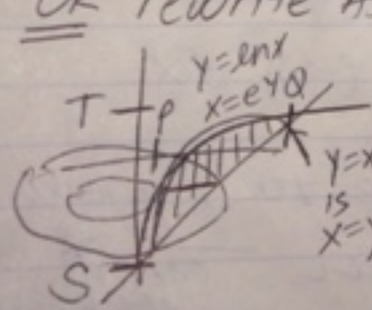


$h = \overset{\text{TOP}}{\ln x} - \overset{\text{BOTTOM}}{(x-2)}$

$r = x$

$$\boxed{V = 2\pi \int_P^Q x \left(\ln x - (x-2) \right) dx}$$

OR rewrite AS $x =$ instead of $y =$



FIND y-values AT points of intersection...
since $y = x - 2$
$T = Q - 2$
$S = P - 2$

NOW IT'S A washer (continued on next page)

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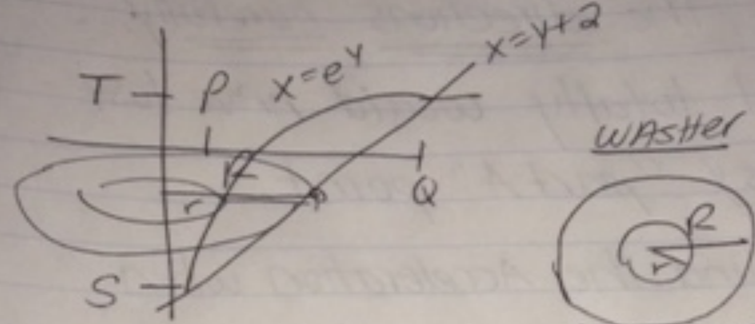
FR5 (A) (cont.)



$S = P-2$

$T = Q-2$

$R = \underset{\text{right}}{(y+2)} - \underset{\text{left}}{\phi} = y+2$

$r = e^y - \phi = e^y$

$$\boxed{V = \pi \int_{P-2}^{Q-2} (y+2)^2 - (e^y)^2 \, dy}$$

Also an ok answer.

So, either...

Shells or **Washers**

$$V = 2\pi \int_P^Q x(\ln x - (x-2))\,dx$$

$$V = \pi \int_{P-2}^{Q-2} (y+2)^2 - (e^y)^2 \, dy$$

are totally OK

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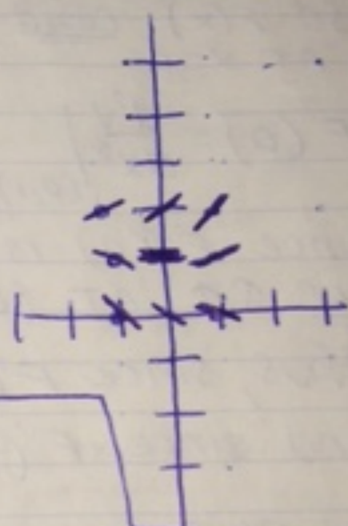
FR 6 – NON-calculator

$\frac{dy}{dx} = \frac{1}{2}x + y - 1$

* Note that this is NON-separable (since x-stuff is added or subtracted to y-stuff so you CAN'T be asked to solve it at your calc level...

(A) slope field

x	y	$m = \frac{1}{2}x + y - 1$
-1	0	$m = -1/2 + 0 - 1 = -\frac{3}{2}$
-1	1	$m = -\frac{1}{2} + 1 - 1 = -1/2$
-1	2	$m = -1/2 + 2 - 1 = 1/2$
0	0	$m = 0 + 0 - 1 = -1$
0	1	$m = 0 + 1 - 1 = 0$
0	2	$m = 0 + 2 - 1 = 1$
1	0	$m = 1/2 + 0 - 1 = -1/2$
1	1	$m = 1/2 + 1 - 1 = 1/2$
1	2	$m = 1/2 + 2 - 1 = 3/2$



(B)

$\frac{dy}{dx} = \frac{1}{2}x + y - 1$

$\frac{d^2y}{dx^2} = \frac{1}{2} + \frac{dy}{dx} = \frac{1}{2} + \left(\frac{1}{2}x + y - 1\right)$

$\boxed{\frac{d^2y}{dx^2} = -\frac{1}{2} + \frac{1}{2}x + y}$

Concave up if

$\frac{d^2y}{dx^2} > 0$

$-\frac{1}{2} + \frac{1}{2}x + y > 0$

$\boxed{y > -\frac{1}{2}x + \frac{1}{2}}$

So concave up anywhere ABOVE the line $y = -\frac{1}{2}x + 1/2$

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FRQ

(C) $f(0)=1$ (again, not that you don't know enough to actually find $y=F(x)$ b/c it's non-separable diff EQ)

Does $F(x)$ have min, max, or neither at $x=0$??

① $F'(0) = \left.\frac{dy}{dx}\right|_{(0,1)} = \frac{1}{2}(0)+1-1 = 0$

so $F(x)$ could have an extrema at $x=0$

② $F''(0) = \left.\frac{d^2y}{dx^2}\right|_{(0,1)} = -\frac{1}{2}+\frac{1}{2}(0)+1 = -\frac{1}{2}+1 = \frac{1}{2}$

since $F''(0)$ is ⊕, $F(x)$ is concave up ∪ at $(0,1)$

thus, since $F'(0)=0$, $F(0)$ is level

and since $F''(0)>0$, $F(0)$ is concave up

so $F(0)=1$ is a $\boxed{\text{minimum}}$

(D) Again, you can't solve this diff EQ... (it's non-separable) but if you are given that $y=mx+b$ is a soln, you can find m and b with the info you have...

continued on next page..

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FR6 D

If $y = mx+b$ is the soln to the DIFF EQ... we know

1. y can be replaced with $mx+b$
2. $\frac{dy}{dx} = m$, so $\frac{dy}{dx}$ can be replaced with m.

thus

$\frac{dy}{dx} = \frac{1}{2}x + y - 1$ can be rewritten as...

$$\downarrow \quad \downarrow \quad \downarrow$$

$$m = \frac{1}{2}x + \underbrace{mx+b}_{} - 1$$

$$m = \underbrace{\left(\frac{1}{2}+m\right)x}_{x\text{ term}} + \underbrace{b-1}_{\text{constant term}}$$

LHS has no x term... thus

$$\underset{\text{LHS}}{0} = \underset{\text{RHS}}{\frac{1}{2}+m} \leftarrow \text{x terms}$$

$$\boxed{-\frac{1}{2} = m}$$

LHS has m as constant... thus

$$\underset{\text{LHS}}{m} = \underset{\text{RHS}}{b-1}$$

$$-\frac{1}{2} = b-1$$

$$\boxed{\frac{1}{2} = b}$$

continued on next page...

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FR6 D (continued)

Another way to do this ...

we know $\frac{dy}{dx} = m$

$m = \frac{1}{2}x + y - 1$

Solve for y ...

$y = m + 1 - \frac{1}{2}x$ rewrite in slope-intercept form

$y = -\frac{1}{2}x + (m+1)$

slope is coefficient of x (called m) ...

intercept is the constant (no x) so $b = m+1$

thus

$\boxed{-\frac{1}{2} = m}$

$b = -\frac{1}{2} + 1$

$\boxed{b = 1/2}$

either method works to get m and b.

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Calculator section (MC #1-4, FR 5)

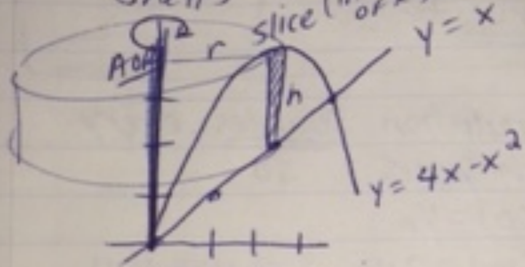
MC1 $y=x$, $y=4x-x^2$ revolved about $\underline{\underline{y\text{-axis}}}$

① Find bounds

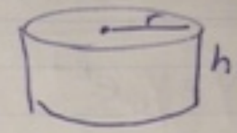
$$x=4x-x^2$$
$$0=3x-x^2$$
$$0=x(3-x)$$
$$x=0 \quad x=3$$

or don't... b/c these are the bounds on $\underline{\text{all}}$ $\underline{\text{answers}}$ (oops)

② b/c AOR // slice, this is Shells



shell:



$r=x$

$h=\underbrace{4x-x^2}_{\text{top}}-\underbrace{x}_{\text{bottom}}$

$$V=\int_0^3 2\pi r h\,dx = 2\pi\int_0^3 x\,(4x-x^2-x)\,dx$$

$$=2\pi\int_0^3 x\,(3x-x^2)\,dx = \boxed{2\pi\int_0^3 (3x^2-x^3)\,dx} \quad \text{E}$$

* NOTE: A calculator doesn't really help here... which is somewhat typical.. there are primarily $\underline{\text{two}}$ types of "calculator section" problems...

$\underline{\text{Type one}}$: A calculator is not useful

→ HINTS: lots of letters as numbers (a, k, b, etc) $\underline{\underline{\text{or}}}$ ans are integrals that aren't evaluated

$\underline{\text{Type two}}$: use your calc!! HINTS: ANS are decimals, there are no extra letters in problem

Ch. 6/7 review solns - APAB

Calculator section (MC1-4, FR5)

MC2 $\frac{dy}{dt} = ky$ Always integrates into

$$y = y_0 e^{kt}$$

(you can do all the steps to get this...

$$\frac{dy}{y} = k\,dt \rightarrow \int \frac{dy}{y} = \int k\,dt \rightarrow \ln|y| = kt + c$$

$$\rightarrow y = e^{kt+c} \rightarrow y = C_1 e^{kt} \rightarrow y = y_0 e^{kt}$$

OR just memorize it...)

$y = y_0 e^{kt}$ population Doubles every ten years... so

$y(0) = y_0$

$y(10) = 2y_0$ ← use this point to find k.

when T=10

$2y_0 = y_0 e^{10k}$

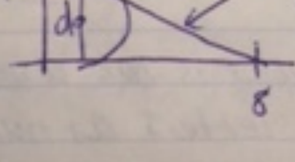
$2 = e^{10k}$

$\ln 2 = 10k$

$k = \frac{\ln 2}{10} \approx \boxed{.069}$

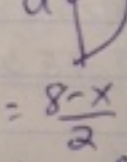
A

MC3



$x + 2y = 8$

$y = \frac{8-x}{2}$



$d = \frac{8-x}{2}$

$r = \frac{d}{2} = \frac{8-x}{4}$

continued on next page...

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MC3 (cont.)

$r = \frac{8-x}{4}$

$A = \frac{1}{2}\pi r^2 \leftarrow$ semicircle

$A = \frac{1}{2}\pi \left(\frac{8-x}{4}\right)^2$

$V = \int_0^8 \frac{1}{2}\pi \left(\frac{8-x}{4}\right)^2 dx$

$Y_1 = (8-x)/4$

$(\pi/2)$ FNINT$(Y_1^2, x, 0, 8)$

$\boxed{V \approx 16.755}$

C

MC4 $0 \le k \le \pi/2$

Area under $y = \cos x$ from $x = k$ to $x = \pi/2$ is .1, find k

Area

$\int_k^{\pi/2} \cos x \, dx = .1$

$[\sin x]_k^{\pi/2} = .1$

$\sin\frac{\pi}{2} - \sin k = .1$

$1 - \sin k = .1$

$-\sin k = -.9$

$\sin k = .9$

now use calculator

$\arcsin(.9) = k$

$\sin^{-1}(.9) = k$

$1.119 \approx k$

(20)

$\boxed{1.120 \approx k}$

D

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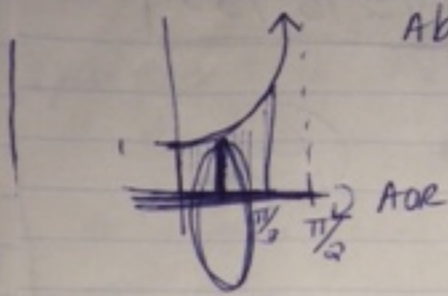
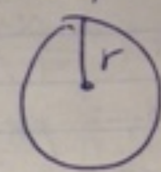
NON-Calculator section (FR6, MC7-10)

MC7 $y = \sec x$, $x = 0$, $y = 0$, $x = \pi/3$



About x-AXIS ... slice ⊥ to AOR
SO IT'S DISKS/wAshers

DISK:



$r = \sec x - 0 = \sec x$

$A = \pi r^2 = \pi \sec^2 x$

$V = \int_0^{\pi/3} \pi \sec^2 x \, dx = \pi \left[\tan x\right]_0^{\pi/3}$

$\tan\frac{\pi}{3} = \frac{\sqrt{3}/2}{1/2} = \sqrt{3}$

$\tan 0 = \frac{0}{1} = 0$

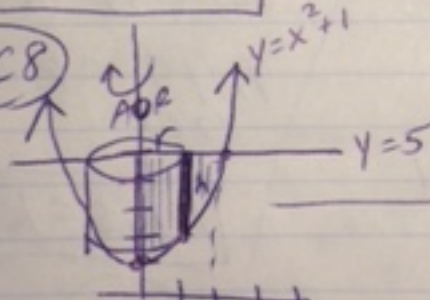
$= \pi \tan\frac{\pi}{3} - \pi \tan 0$

$= \pi\sqrt{3} - \pi(0)$

$= \boxed{\pi\sqrt{3}}$

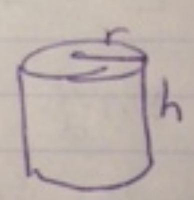
C

MC8



AOR || to slice SO
SHELLS

Shell



$r = x$

$h = 5 - (x^2+1) = 5 - x^2 - 1$
TOP BOTTOM
$= 4 - x^2$

$A = 2\pi r h$
$= 2\pi x(4 - x^2)$
$= 2\pi(4x - x^3)$

$x^2 + 1 = 5$
$x^2 = 4$
$x = \pm 2$

$V = 2\pi \int_0^2 4x - x^3 \, dx$

$V = 2\pi \left[2x^2 - \frac{x^4}{4}\right]_0^2 = 2\pi \left[\left(8 - \frac{16}{4}\right) - \varnothing\right]$

$= 2\pi[4] = \boxed{8\pi}$

B

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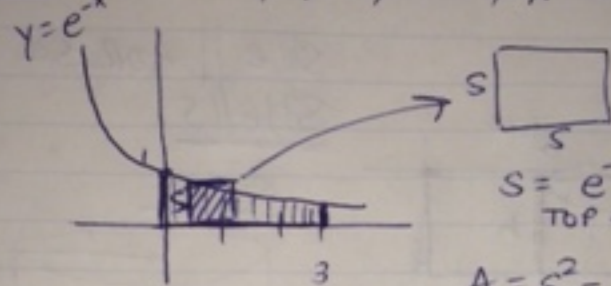
non-calculator section (FR6, MC7-10)

MC9 $y = e^{-x}$, $\overbrace{y=0, x=0}^{\text{Axes}}$, $x = 3$



$s = e^{-x} - 0 = e^{-x}$
TOP - BOTTOM

$A = s^2 = (e^{-x})^2 = e^{-2x}$

$$V = \int_0^3 e^{-2x}\,dx$$

$$= \left[-\tfrac{1}{2}e^{-2x}\right]_0^3$$

$$= -\tfrac{1}{2}e^{-6} - \left(-\tfrac{1}{2}e^{0}\right)$$

$$= -\tfrac{1}{2}e^{-6} + \tfrac{1}{2}$$

$$= \boxed{\frac{1-e^{-6}}{2}} \quad A$$

u-sub work

$\int e^{-2x}dx$

$u = -2x$

$du = -2dx$

$\frac{du}{-2} = dx$

$\int e^u \cdot \frac{du}{-2} = \frac{-1}{2}\int e^u du$

$= -\frac{1}{2}[e^u] = \left[-\frac{1}{2}e^{-2x}\right]$

super common error $e^0 = 1$

$-\frac{1}{2}e^0 = -\frac{1}{2}(1) = -\frac{1}{2}$

the common mistake is people think it's $(-\frac{1}{2}e)^0 = 1$ but the $-\frac{1}{2}$ is NOT inside the 0 power!!

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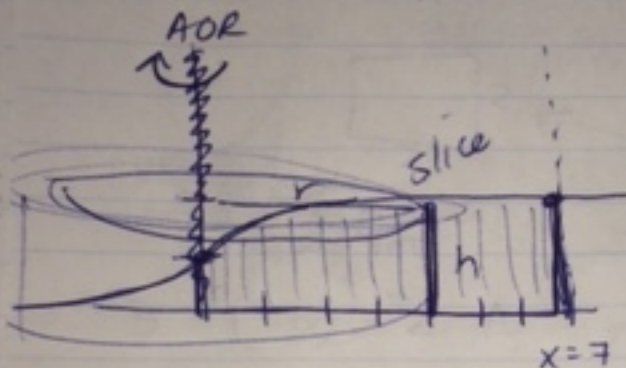
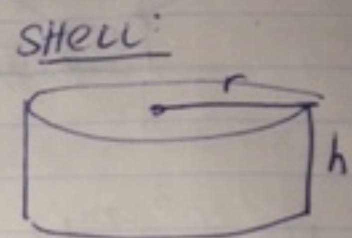
NON-CAlculator section (FR6, MC7-10)

MC10 $y = (x+1)^{1/3}$, $x = 7$, $\underbrace{x=0, y=0}_{\text{Axes}}$



slice || AOR so

SHells



$V = \int_0^7 2\pi x (x+1)^{1/3} dx$

$= \boxed{2\pi \int_0^7 x (x+1)^{1/3} dx}$

B

$r = x$

$h = (x+1)^{1/3} - \emptyset$

TOP - BOTTOM

$h = (x+1)^{1/3}$

$A = 2\pi r h$

$A = 2\pi x \, (x+1)^{1/3}$

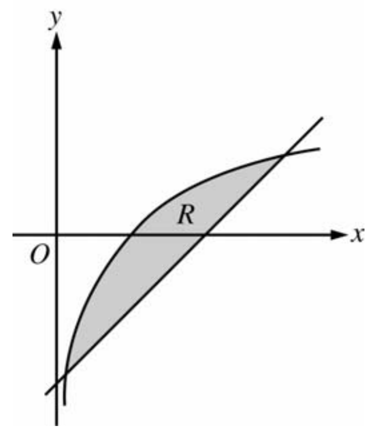
Question 1

Let R be the shaded region bounded by the graph of $y = \ln x$ and the line $y = x - 2$, as shown above.

(a) Find the area of R.

(b) Find the volume of the solid generated when R is rotated about the horizontal line $y = -3$.

(c) Write, but do not evaluate, an integral expression that can be used to find the volume of the solid generated when R is rotated about the y-axis.



$\ln(x) = x - 2$ when $x = 0.15859$ and 3.14619.
Let $S = 0.15859$ and $T = 3.14619$

(a) Area of $R = \int_S^T (\ln(x) - (x - 2))\, dx = 1.949$

$3 : \begin{cases} 1 : \text{integrand} \\ 1 : \text{limits} \\ 1 : \text{answer} \end{cases}$

(b) Volume $= \pi \int_S^T \left((\ln(x) + 3)^2 - (x - 2 + 3)^2 \right) dx$
$= 34.198$ or 34.199

$3 : \begin{cases} 2 : \text{integrand} \\ 1 : \text{limits, constant, and answer} \end{cases}$

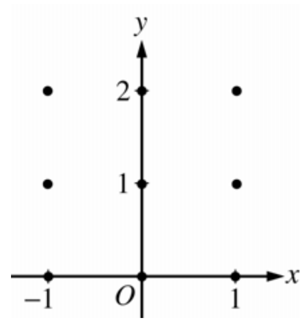
(c) Volume $= \pi \int_{S-2}^{T-2} \left((y + 2)^2 - \left(e^y\right)^2 \right) dy$

$3 : \begin{cases} 2 : \text{integrand} \\ 1 : \text{limits and constant} \end{cases}$

Question 5

Consider the differential equation $\frac{dy}{dx} = \frac{1}{2}x + y - 1$.

(a) On the axes provided, sketch a slope field for the given differential equation at the nine points indicated.
(Note: Use the axes provided in the exam booklet.)

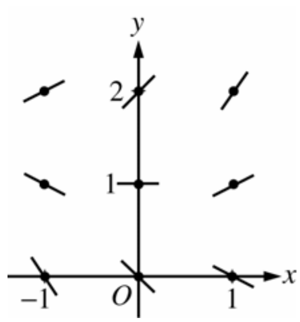


(b) Find $\frac{d^2y}{dx^2}$ in terms of x and y. Describe the region in the xy-plane in which all solution curves to the differential equation are concave up.

(c) Let $y = f(x)$ be a particular solution to the differential equation with the initial condition $f(0) = 1$. Does f have a relative minimum, a relative maximum, or neither at $x = 0$? Justify your answer.

(d) Find the values of the constants m and b, for which $y = mx + b$ is a solution to the differential equation.

(a)



2 : Sign of slope at each point and relative steepness of slope lines in rows and columns.

(b) $\frac{d^2y}{dx^2} = \frac{1}{2} + \frac{dy}{dx} = \frac{1}{2}x + y - \frac{1}{2}$

Solution curves will be concave up on the half-plane above the line $y = -\frac{1}{2}x + \frac{1}{2}$.

$3 : \begin{cases} 2 : \frac{d^2y}{dx^2} \\ 1 : \text{description} \end{cases}$

(c) $\left.\frac{dy}{dx}\right|_{(0,1)} = 0 + 1 - 1 = 0$ and $\left.\frac{d^2y}{dx^2}\right|_{(0,1)} = 0 + 1 - \frac{1}{2} > 0$

Thus, f has a relative minimum at $(0, 1)$.

$2 : \begin{cases} 1 : \text{answer} \\ 1 : \text{justification} \end{cases}$

(d) Substituting $y = mx + b$ into the differential equation:

$m = \frac{1}{2}x + (mx + b) - 1 = \left(m + \frac{1}{2}\right)x + (b - 1)$

Then $0 = m + \frac{1}{2}$ and $m = b - 1$: $m = -\frac{1}{2}$ and $b = \frac{1}{2}$.

$2 : \begin{cases} 1 : \text{value for } m \\ 1 : \text{value for } b \end{cases}$